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# NAVAL POSTGRADUATE SCHOOL

## Monterey, California



## THESIS

MINIMUM TIME CONTROL OF  
A THIRD ORDER REGULATOR

by

Kayhan Vardareri

December, 1991

Thesis Advisor:

Harold A. Titus

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The optimal minimum time control is applied to a third order regulator. From Pontryagin, the optimal control must minimize the Hamiltonian. The control is a function of the states. The state space is partitioned into regions where the optimal control is either plus or minus N (the maximum control effort) which is bang-bang control.

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Minimum Time Control of  
A Third Order Regulator

by

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Submitted in partial fulfillment  
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## ABSTRACT

The optimal minimum time control is applied to a third order regulator. From Pontryagin, the optimal control must minimize the Hamiltonian. The control is a function of the states. The state space is partitioned into regions where the optimal control is either plus or minus  $N$  (the maximum control magnitude) which is bang-bang control.

c.1

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## I. INTRODUCTION

In recent years, much attention has been focused upon optimizing the behavior of systems. Some particular problems are maximizing the range of a rocket, minimizing the error in estimation of position of an object, and minimizing the time required to reach some required final state. To solve an optimization problem, we must first define a goal or a performance function for the system we are trying to optimize. Once we have chosen the performance function, we may determine the optimal control which minimizes (or maximizes) the performance function.

Optimization has some disadvantages besides its advantages. The mathematical formulation of the design requirement is sometimes not that easy. For high-order systems, especially, ( $n \geq 2$ ) it is usually difficult to determine an analytical expression for the performance function. An optimal system may be very sensitive to wrong starting assumptions and a system optimal from one point may not be optimal from another point.

In this report, we will develop a minimum-time control for a third order regulator. Our objective is to determine a control as a function of the states that transfers the system



from an arbitrary initial state to a specified final state  
(which is the origin) in minimum time.

## II. MINIMUM TIME CONTROL

The objective in optimal control problems is to determine a control that minimizes the performance function.

### A. PONTRYAGIN'S PRINCIPLE

The principle of Pontryagin is of considerable importance in the theory of optimum control systems. First, let's formulate Pontryagin's principle. The state equations of a linear, time-invariant, controlled system of order  $n$  having  $m$  controls are given as

$$\dot{x}(t) = Ax(t) + Bu(t) \quad (2.1)$$

$$y(t) = Cx(t) + Du(t) \quad (2.2)$$

where  $A$  and  $B$  are constant  $n \times n$  and  $n \times m$  matrices, respectively. The control vector  $u(t)$  is required to be piece-wise continuous and has to be bounded.

We can define the optimal control as follows. The state of the system is defined by the initial condition  $x(t_0)$  at the initial time  $t_0$ . The system is to be transferred to the final state  $x(t_f)$  by an admissible control  $u(t)$ , where

$$J = \int_{t_0}^{t_f} dt = t_f - t_0 \quad (2.3)$$

the performance of the system is minimum. The trajectory  $x^*(t)$  generated by the optimal control is called the optimal state trajectory. This trajectory is the solution of the vector differential equations (2.1) and (2.2) forced by the optimal control,  $u^*(t)$ . The system is time-invariant and  $x(t_0)$  and  $x(t_f)$  are fixed. The problem is the optimal control of a time-invariant system with fixed boundary points. The Hamiltonian is

$$H(x(t), u(t), p(t), t) = 1 + p^T(t)Ax(t) + p^T(t)Bu(t) \quad (2.4)$$

where  $p(t)$  is the Lagrange multiplier.

The necessary condition for  $u^*(t)$  to minimize the performance function  $J$  is

$$H(x^*(t), u^*(t), p^*(t), t) \leq H(x^*(t), u(t), p^*(t), t) \quad (2.5)$$

or in other words

$$1 + p^{*T}Ax^*(t) + p^{*T}Bu^*(t) \leq 1 + p^{*T}(t)Ax^*(t) + p^{*T}(t)Bu(t) \quad (2.6)$$

for all  $t \in [t_0, t_f]$  and for all admissible controls.

The equations (2.5) and (2.6) are called Pontryagin's minimum principle. The necessary conditions for  $u^*(t)$  to be the optimal control are

$$\dot{x}^*(t) = \frac{\partial H}{\partial p}(x^*(t), u^*(t), p^*(t), t) \quad (2.7)$$

$$\dot{p}^*(t) = -\frac{\partial H}{\partial x}(x^*(t), u^*(t), p^*(t), t) \quad (2.8)$$

$$H(x^*(t), u^*(t), p^*(t), t) \leq H(x^*(t), u(t), p^*(t), t) \quad (2.9)$$

for all  $t \in [t_0, t_f]$ .

## B. MINIMUM TIME SYSTEMS

Here, the only measure of performance is the minimization of the transition time from an arbitrary initial state to the final state. Mathematically, then, our problem is to transfer a system given by the equations (2.1) and (2.2) from an arbitrary initial state to the desired final state and minimize the performance function that is described by the equation (2.3).

The control is bounded

$$|u_i| \leq N \quad i = 1, 2, \dots, m \quad (2.10)$$

The scalar control variable is either  $u=+N$  or  $u=-N$ , and the signs alternate. This is called the bang-bang principle. That is the optimal control switches between its maximum and minimum admissible values.



### C. OPTIMAL CONTROL OF LINEAR, TIME-INVARIANT CONTROLLED SYSTEMS

The controlled system given in equations (2.1) and (2.2) is assumed to be observable, controllable. The time-invariant system is controllable if and only if the rank  $r(Q)$  of the controllability test matrix

$$Q = [B \ AB \dots A^{n-1}B] \quad (2.11)$$

is equal to  $n$ , the order of the system. [Ref.2]

The time-invariant system is observable if and only if the rank  $r(R)$  of the observability test matrix

$$R = [C^T \ A^T C^T \dots (A^T)^{n-1} C^T] \quad (2.12)$$

is equal to  $n$ , the order of the system. [Ref.2]

In equation (2.4), the control vector  $u$  occurs only in the last term of the Hamiltonian so that, only this term is to be minimized

$$p^{*T} B u_i(t) \quad (2.13)$$

When the coefficient of  $u_i(t)$  is positive,  $u_i^*(t)$  must be the smallest admissible control  $-N$  and when the coefficient of  $u_i(t)$  is negative,  $u_i^*(t)$  must be the biggest admissible control  $+N$ .

$$u_i^*(t) = -N \operatorname{sign}(p^{*T} B) \quad (2.14)$$

This equation is known as the mathematical statement of the bang-bang principle.

### 1. Existence Of The Minimum-Time Control

Finding a control, if one exists, that transfers the system from an arbitrary initial state  $x(t_0)$  to a desired state  $x(t_f)=0$  in minimum time, is called the linear regulator, minimum-time problem. Before trying to find an optimal control, there are three theorems due to Pontryagin that we need to check.

1. If all of the eigenvalues of  $A$  have nonpositive real parts, then an optimal control exists.

2. If an extremal control exists, then it is unique.

A control that satisfies the conditions in equations (2.7), (2.8), and (2.9) is an extremal control.

3. If all the eigenvalues of  $A$  are real and an unique time-optimal control exists, then each control component can switch at most  $(n-1)$  times.

### III. SECOND ORDER CONTROLLER

#### A. MINIMUM TIME OPTIMAL CONTROL

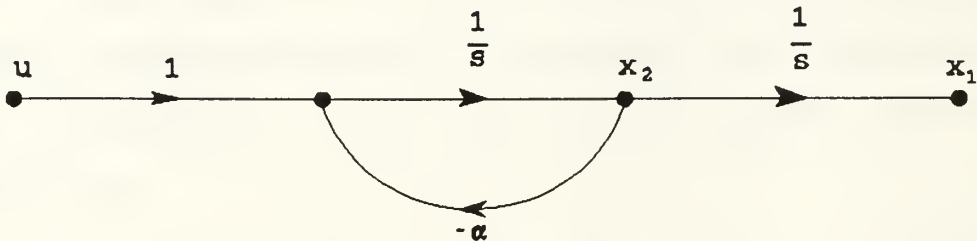
When we specify information concerning the desired states at the final time and the initial condition vector, we have a two-point boundary value problem with half of the conditions specified at the initial time and the other half specified at the final time. A possible method of solution is reversing time in the equations; starting at the specified final vector which is often the origin of the state vector with a constant control until a switching point is obtained. [Ref. 2]

##### 1. Problem Definition

Consider the system

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad (3.1)$$

where  $u$  is the control variable bounded:  $|u(t)| \leq N$ . This can be represented in flow diagram as



The transfer function of the system is

$$\frac{X(s)}{U(s)} = G(s) = \frac{1}{s(s+\alpha)} \quad (3.2)$$

The Hamiltonian is

$$H = 1 + p_1 x_2 - \alpha p_2 x_2 + p_2 u \quad (3.3)$$

From Pontryagin [Ref.1], we find we can minimize  $J$  by minimizing the Hamiltonian. This is achieved with

$$u = -N \operatorname{sign}(p_2) \quad (3.4)$$

where

$$\dot{p}_1(t) = -\frac{\partial H}{\partial x_1} = 0 \quad (3.5)$$

$$\dot{p}_2(t) = -\frac{\partial H}{\partial x_2} = -p_1(t) + \alpha p_2(t) \quad (3.6)$$

so that

$$p_1(t) = p_1(0) \quad (3.7)$$

$$p_2(t) = -p_1(0)t + p_2(0)e^{-\alpha t} + p_2(0) \quad (3.8)$$

First, we need to check the controllability and observability of the system.

$$Q = \begin{bmatrix} 0 & 1 \\ 1 & -\alpha \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad (3.9)$$



Both matrices have a rank of 2, so that, the system is controllable, observable.

Since, the eigenvalues of  $A$ , 0 and  $-\alpha$ , are nonpositive real numbers, an optimal control exists, and this extremal control is unique.

Note that, from the theorems due to Pontryagin, the control may change sign only once. There are only two possible paths for the states to take: one corresponding to  $u=+N$  and one corresponding  $u=-N$ .

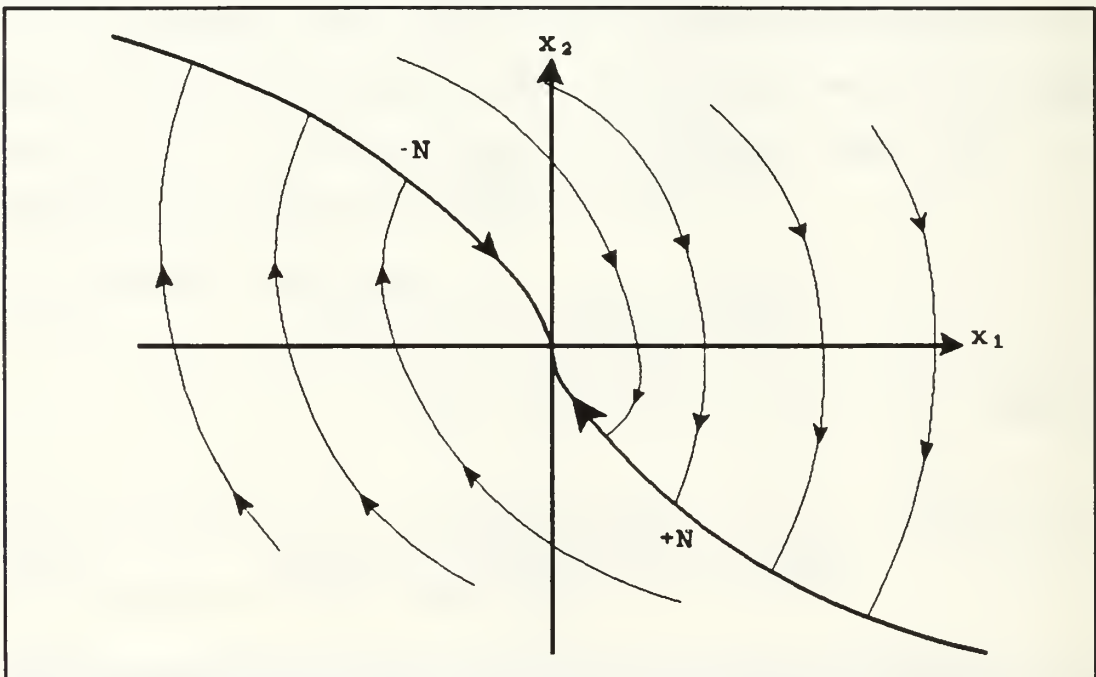


Figure 3.1 Minimum Time Trajectory Solution Curves

## B. SOLUTION TO SECOND ORDER SWITCHING CURVES

We will solve this problem for arbitrary initial conditions,  $x_1(0)$  and  $x_2(0)$ . But, we demand that the final states,  $x_1(t_f)$  and  $x_2(t_f)$ , will be zero.

### 1. Second Order Switching Laws

We will solve the problem by transforming the state equations into an equivalent uncoupled system. First, we define a matrix  $G$  whose columns are the eigenvectors of  $A$ , and define a new dependent variable  $y$  by

$$x = Gy \quad (3.10)$$

Then, substituting for  $x$  in equation (2.1), we obtain

$$Gy = AGy + Bu \quad (3.11)$$

By multiplying by  $G^{-1}$  it follows that

$$y = G^{-1}AGy + G^{-1}Bu \quad (3.12)$$

After a little algebra, we find

$$G = \begin{bmatrix} \frac{1}{\alpha} & -\frac{1}{\alpha} \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad G^{-1} = \begin{bmatrix} \alpha & 1 \\ 0 & 1 \end{bmatrix} \quad (3.13)$$

$$\dot{y} = \begin{bmatrix} \alpha & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & -\alpha \end{bmatrix} \begin{bmatrix} \frac{1}{\alpha} & -\frac{1}{\alpha} \\ 0 & 1 \end{bmatrix} y + \begin{bmatrix} \alpha & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \quad (3.14)$$

$$\dot{y} = \begin{bmatrix} 0 & 0 \\ 0 & -\alpha \end{bmatrix} y + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u \quad (3.15)$$

or in scalar equations

$$\dot{y}_1(t) = u \quad (3.16)$$

$$\dot{y}_2(t) = -\alpha y_2(t) + u \quad (3.17)$$

The forward time solutions of these two equations are

$$y_1(t) = y_1(0) + ut \quad (3.18)$$

$$y_2(t) = y_2(0) e^{-\alpha t} + \frac{u}{\alpha} (1 - e^{-\alpha t}) \quad (3.19)$$

## 2. Negative Time System

In negative time, we start at the specified final state which is the origin with a constant control until a switching point is obtained. In negative time we have

$$\dot{y} = \begin{bmatrix} 0 & 0 \\ 0 & \alpha \end{bmatrix} y + \begin{bmatrix} -1 \\ -1 \end{bmatrix} u \quad (3.20)$$

or in scalar equations

$$\dot{y}_1(t) = -u \quad (3.21)$$

$$\dot{y}_2(t) = \alpha y_2(t) - u \quad (3.22)$$

The solutions of these two first order differential equations are

$$y_1(t) = y_1(0) - ut \quad (3.23)$$

$$y_2(t) = y_2(0) e^{\alpha t} + \frac{u}{\alpha} (1 - e^{\alpha t}) \quad (3.24)$$

By putting  $u=+N$  in the negative time equations we have

$$y_1(t) = y_1(0) - Nt \quad (3.25)$$

$$y_2(t) = y_2(0) e^{\alpha t} + \frac{N}{\alpha} (1 - e^{\alpha t}) \quad (3.26)$$

And with  $u=-N$  we get

$$y_1(t) = y_1(t) + Nt \quad (3.27)$$

$$y_2(t) = y_2(0) e^{\alpha t} - \frac{N}{\alpha} (1 - e^{\alpha t}) \quad (3.28)$$

Since, we are going in negative time, the starting point is the origin, so that  $y_1(0)=0$  and  $y_2(0)=0$ . So, in negative time from the origin for  $u=+N$ , we have

$$y_1(t) = -Nt \quad (3.29)$$

$$y_2(t) = \frac{N}{\alpha} (1 - e^{\alpha t}) \quad (3.30)$$

For  $u=-N$  we have

$$y_1(t) = Nt \quad (3.31)$$

$$y_2(t) = -\frac{N}{\alpha} (1 - e^{\alpha t}) \quad (3.32)$$

By eliminating  $t$  from equations (3.30) and (3.32) and substituting these into equations (3.29) and (3.31) and by taking care of the sign changes due to  $+N$  or  $-N$ , we get

$$u = -N \operatorname{sign} \left( y_1 - \operatorname{sign}(y_2) \frac{N}{\alpha} \ln \left( 1 + \frac{\alpha}{N} |y_2| \right) \right) \quad (3.33)$$

The second order switching curve of equation (3.33) is simulated in Figure 3.2.

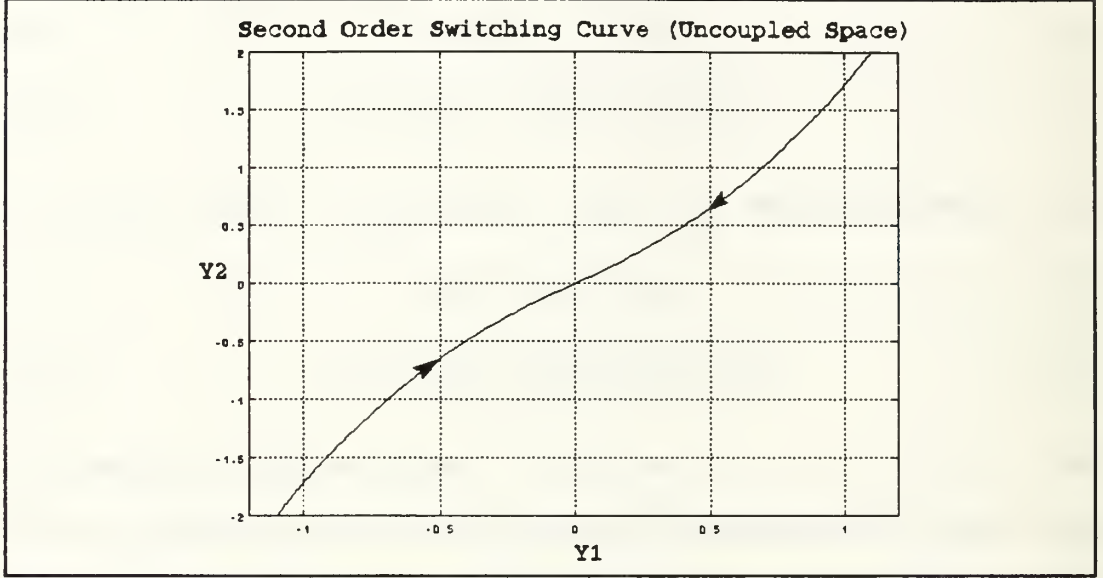


Figure 3.2 The Switching Curve in Uncoupled Space

From equation (3.10) we get

$$y = G^{-1} x \quad (3.34)$$

so that

$$y_1 = \alpha x_1 + x_2 \quad (3.35)$$

$$y_2 = x_2 \quad (3.36)$$

And when we go back to  $x$  variables with equations (3.35) and (3.36), we can define the switching law as

$$u = -N \operatorname{sign} \left( \alpha x_1 + x_2 - \operatorname{sign}(x_2) \frac{N}{\alpha} \ln \left( 1 + \frac{\alpha}{N} |x_2| \right) \right) \quad (3.37)$$

A simulation of this switching law is given in Figure 3.3.

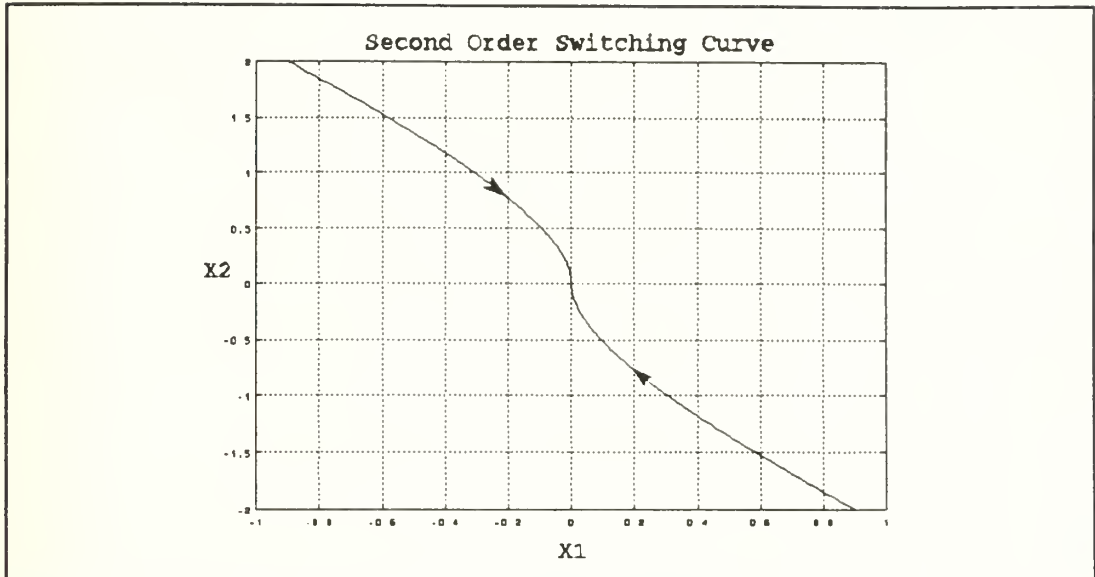


Figure 3.3 The Second Order Switching Curve

We drive the states from an arbitrary initial condition to the origin with this switching law with, at most, one change of the control effort.

### 3. Simulation of the Second Order Control in Uncoupled Space

We need to test the accuracy of the solutions by using a computer solution. To test the switching law of equation (3.33), we simulate the system in uncoupled space using a maximum control effort of  $N=1$  and  $\alpha=1$ . The output of the simulation is shown in Figure 3.4. As we can see from this figure, the control effort which is given in Figure 3.5, drives the states first to the zero trajectory curve, then to the origin. After that, in order to demonstrate the effects of



$N$  and  $\alpha$ , we run the simulation first with a maximum control effort of  $N=5$ . The results of this simulation are given in Figures 3.6 and 3.7. The effect of  $\alpha$  are presented in Figures 3.8 and 3.9.

As we can see from these figures, increasing the magnitude of the control effort shortens the response time of the system. A bigger time constant,  $\alpha$ , slows down the response of the system.

#### 4. Simulation of the Second Order Control

Now, we can simulate the system of equation (3.1) in normal two-dimensional space whose switching law is given by equation (3.37). The same control effort, time constant and initial conditions which are used to simulate the system in uncoupled space are used here again. First, simulation is with a maximum control effort of  $N=1$  and a time constant of  $\alpha=1$ . The results of this simulation are given in Figures 3.10 and 3.11. The second simulation is with a maximum control effort of  $N=5$  whose results are given in Figures 3.12 and 3.13. The last simulation is with a time constant of  $\alpha=5$  to show the effect of the time constant in the system. Results of this simulation are given in Figures 3.14 and 3.15. Again, bigger control effort shortens the response time of the system and bigger time constant slows down the response of the system. After these simulations, in order to test that the switching

laws work for an arbitrary initial condition, the system is simulated with initial conditions  $x_1(0)=-1$ ,  $x_2(0)=-1$ , whose results are given in Figures 3.16 and 3.17, and  $x_1(0)=-1$ ,  $x_2(0)=1$ , whose results are given in Figures 3.18 and 3.19.

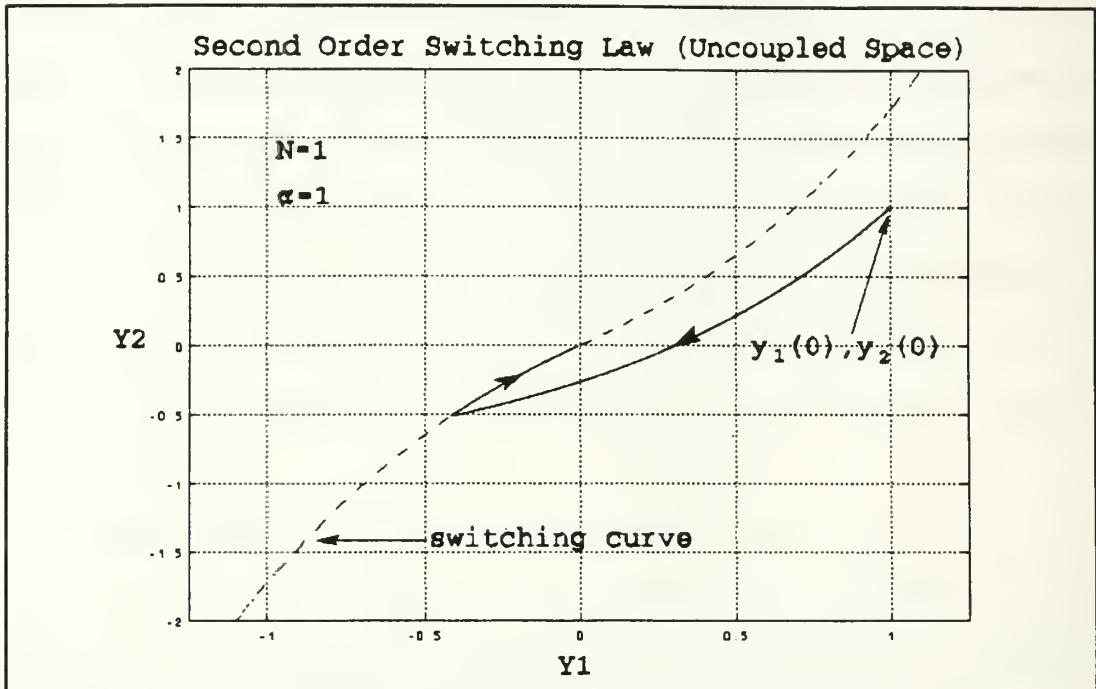


Figure 3.4 Simulation of Second Order Controller

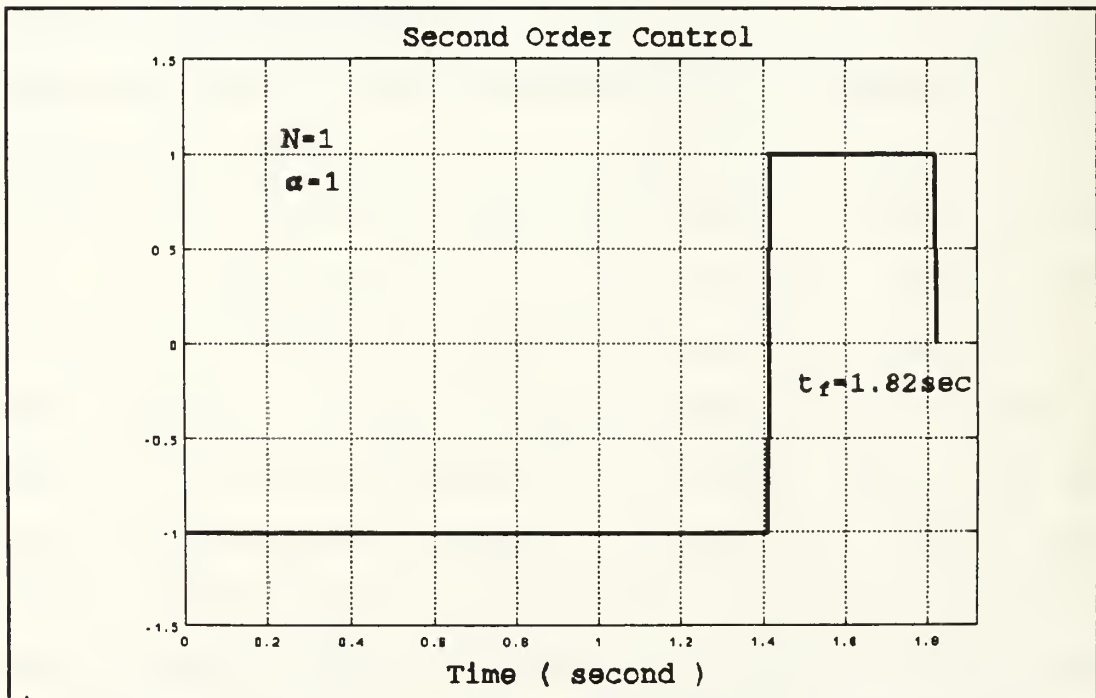


Figure 3.5 Control Effort for Second Order Controller

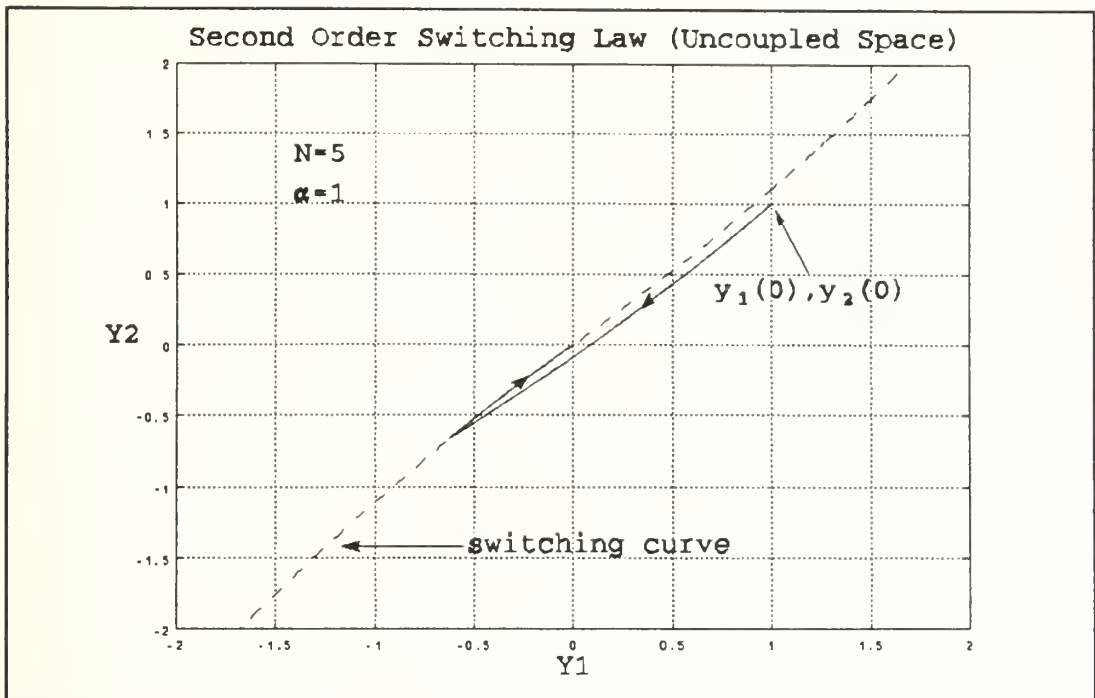


Figure 3.6 Simulation of Second Order Controller

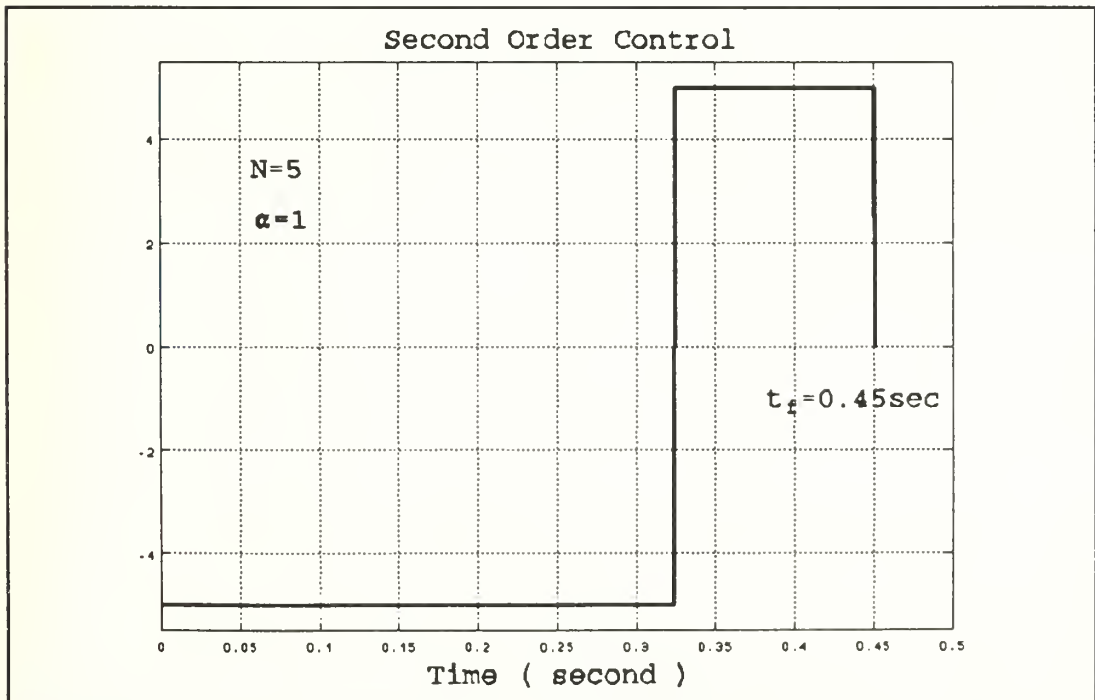


Figure 3.7 Control Effort for Second Order Controller

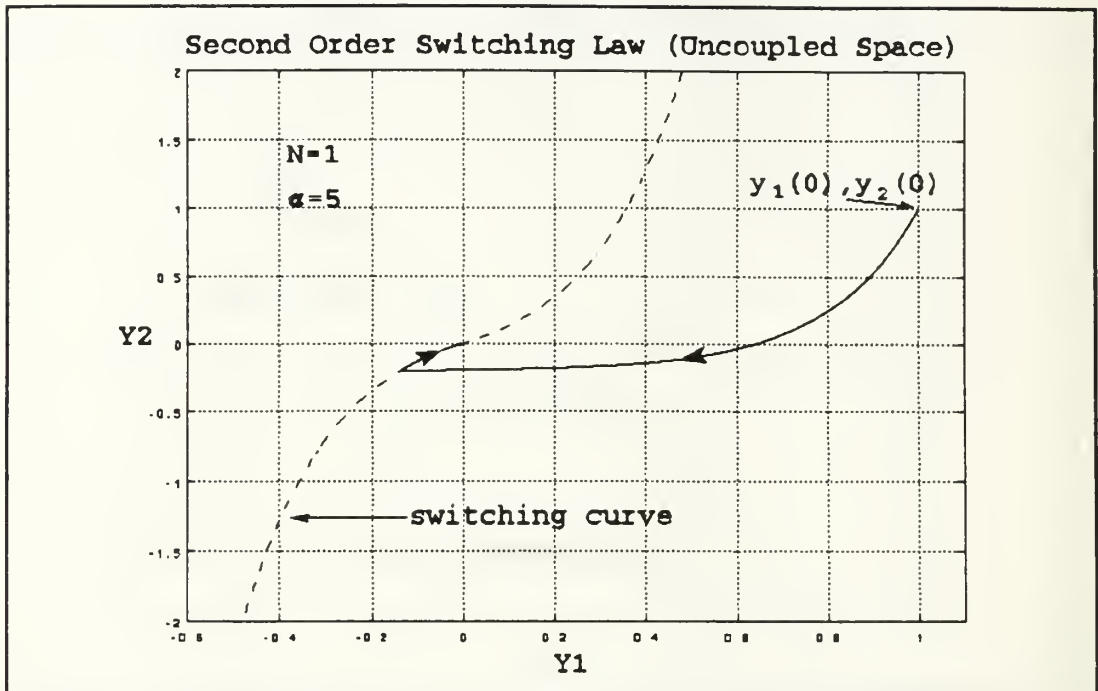


Figure 3.8 Simulation of Second Order Controller

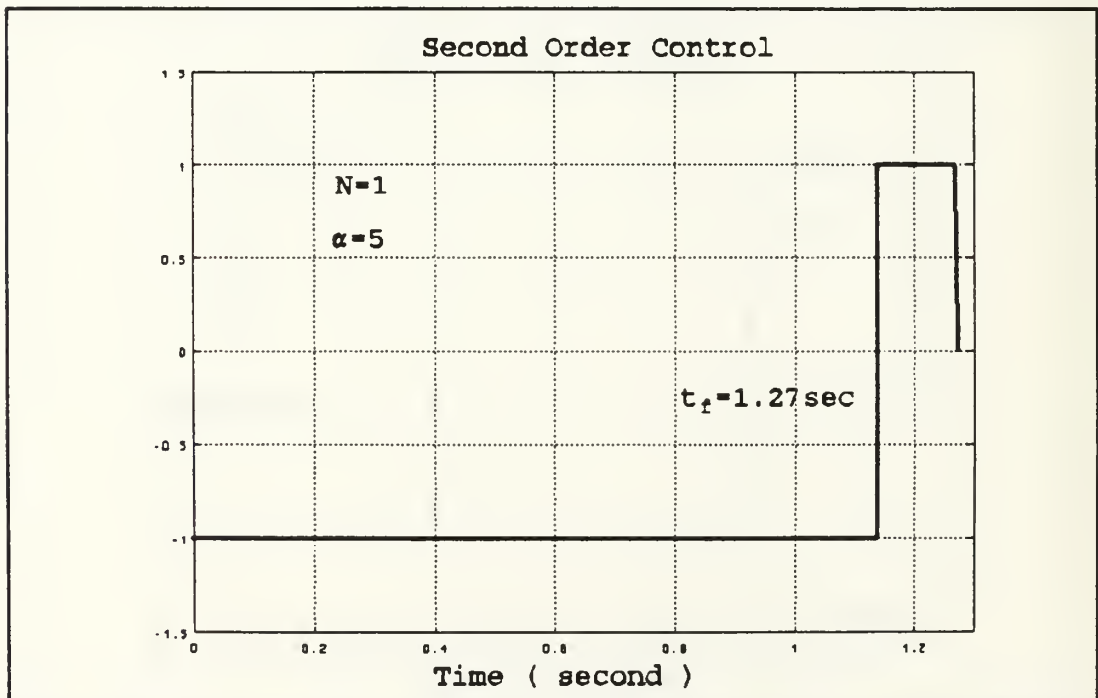


Figure 3.9 Control Effort for Second Order Controller

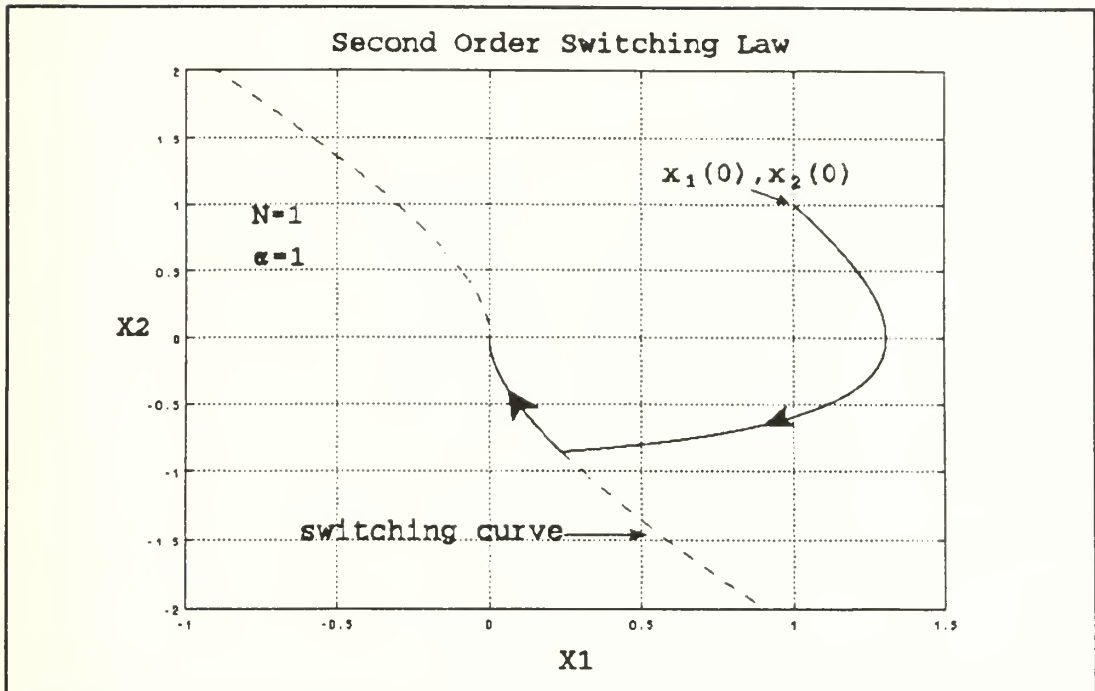


Figure 3.10 Simulation of Second Order Controller

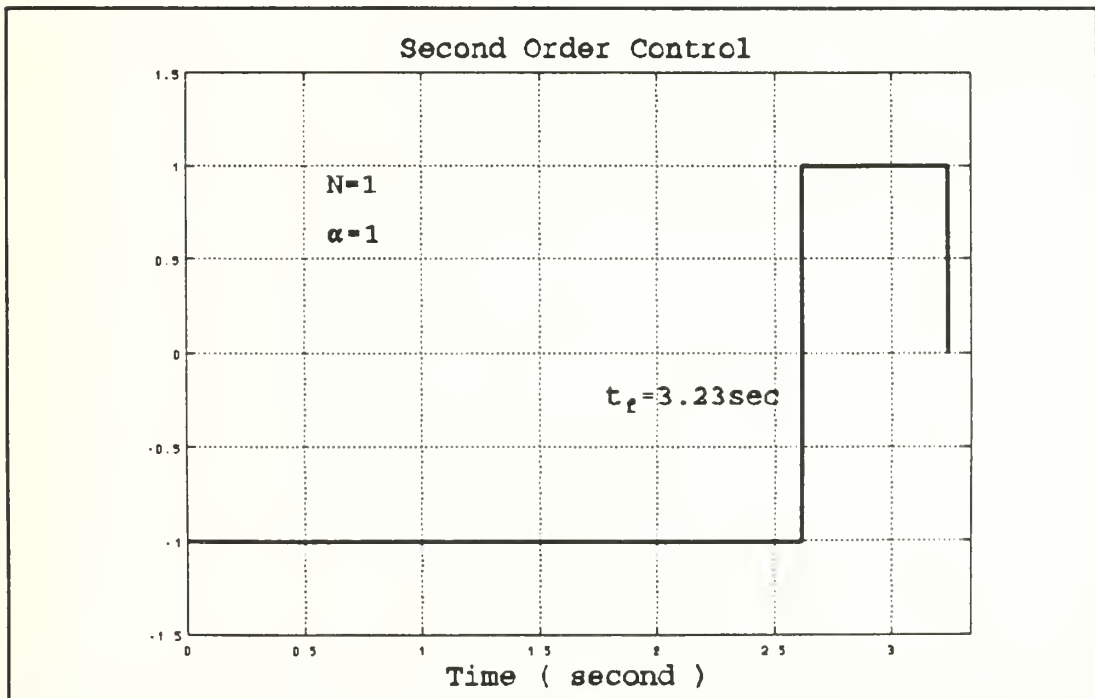


Figure 3.11 Control Effort for Second Order Controller



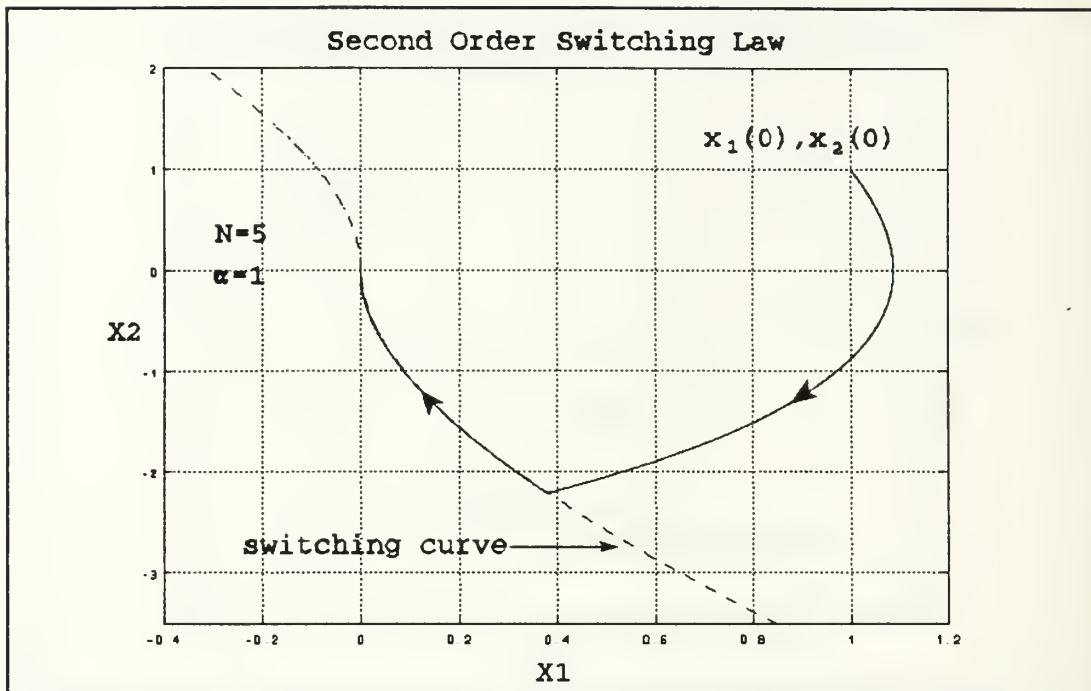


Figure 3.12 Simulation of Second Order Controller

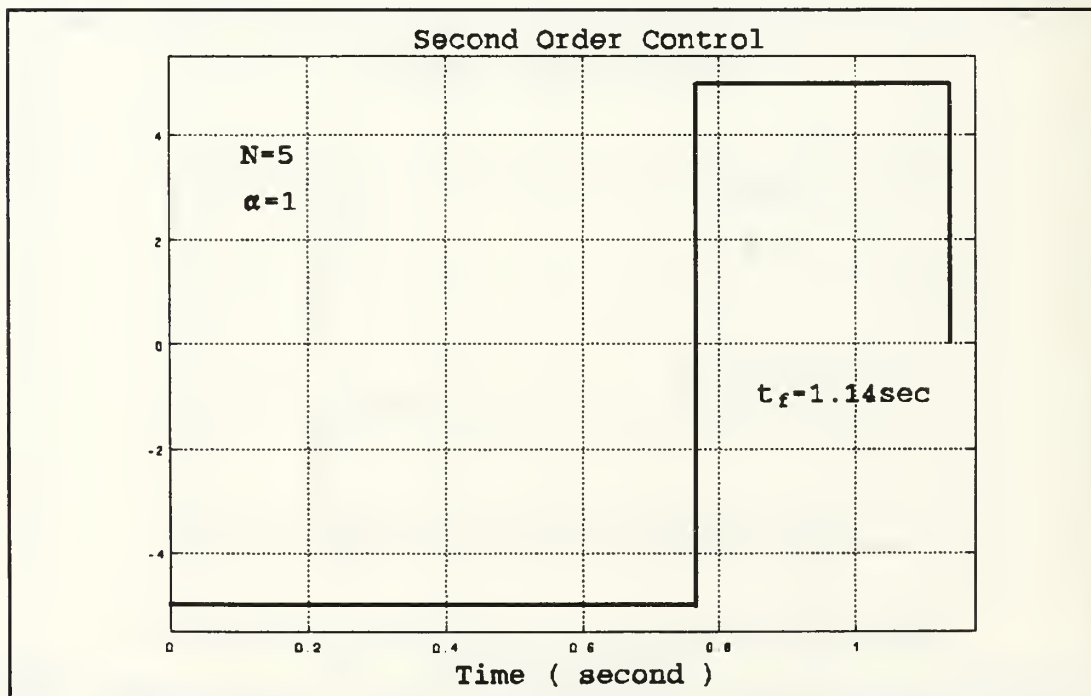


Figure 3.13 Control Effort for Second Order Controller

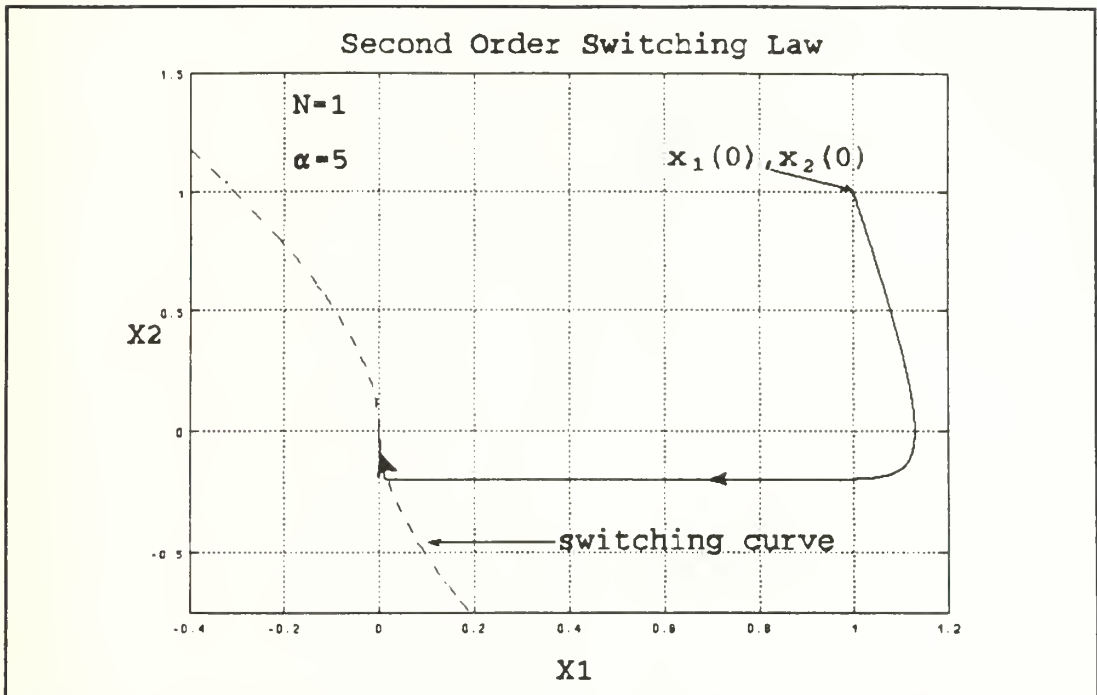


Figure 3.14 Simulation of Second Order Controller

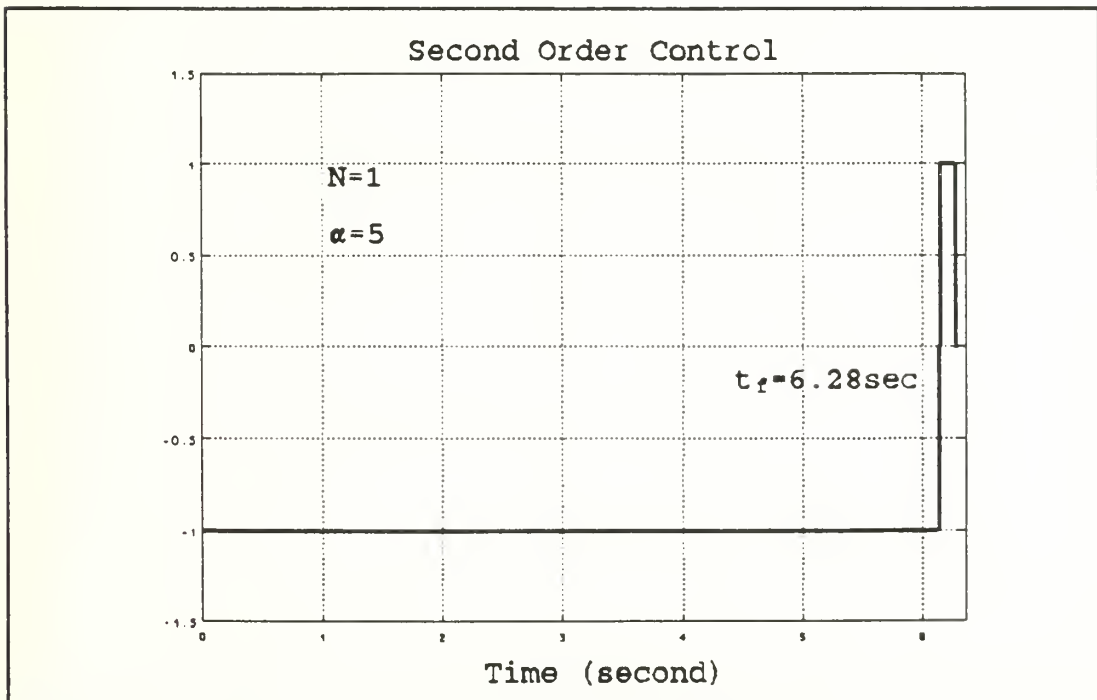


Figure 3.15 Control Effort for Second Order Controller

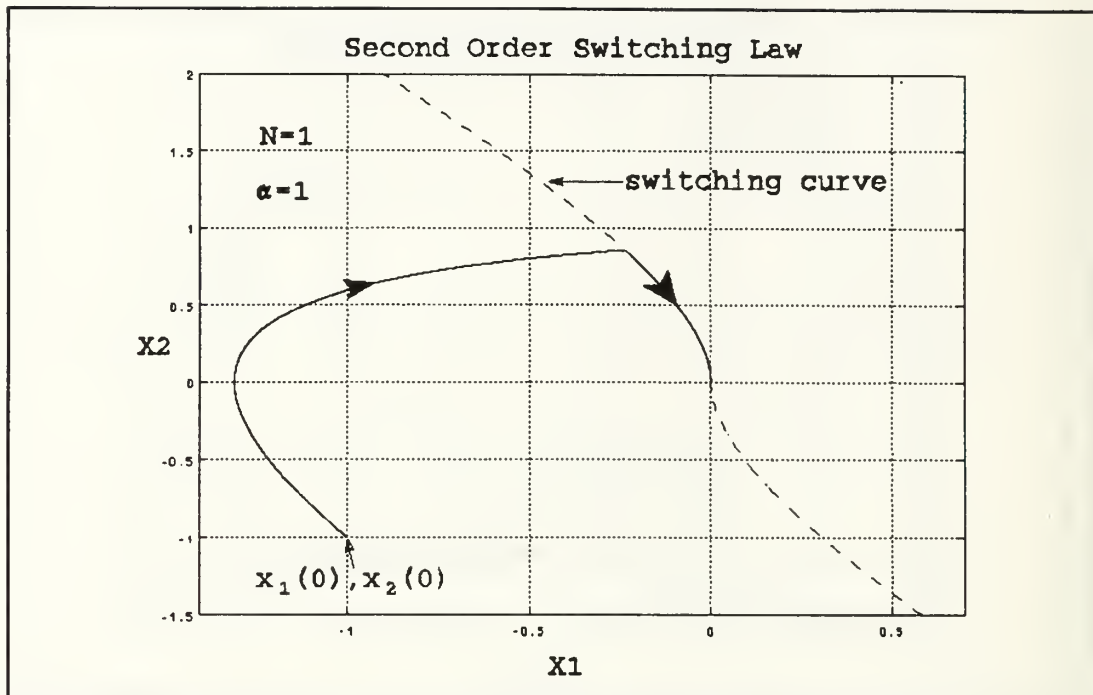


Figure 3.16 Simulation of Second Order Controller

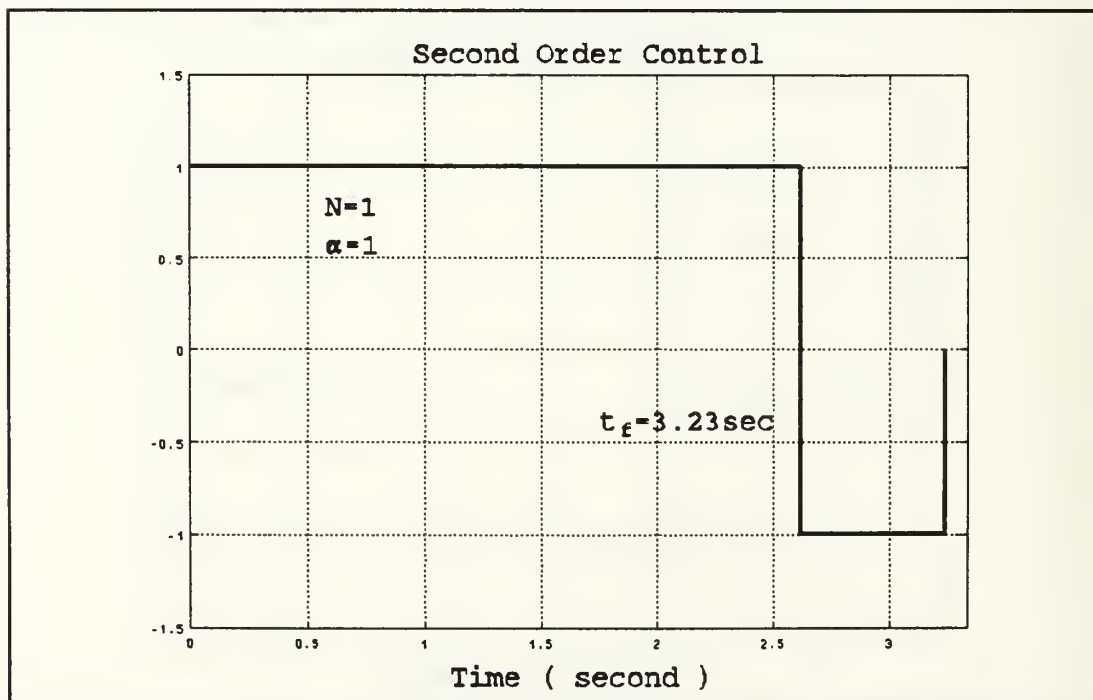


Figure 3.17 Control Effort for Second Order Controller

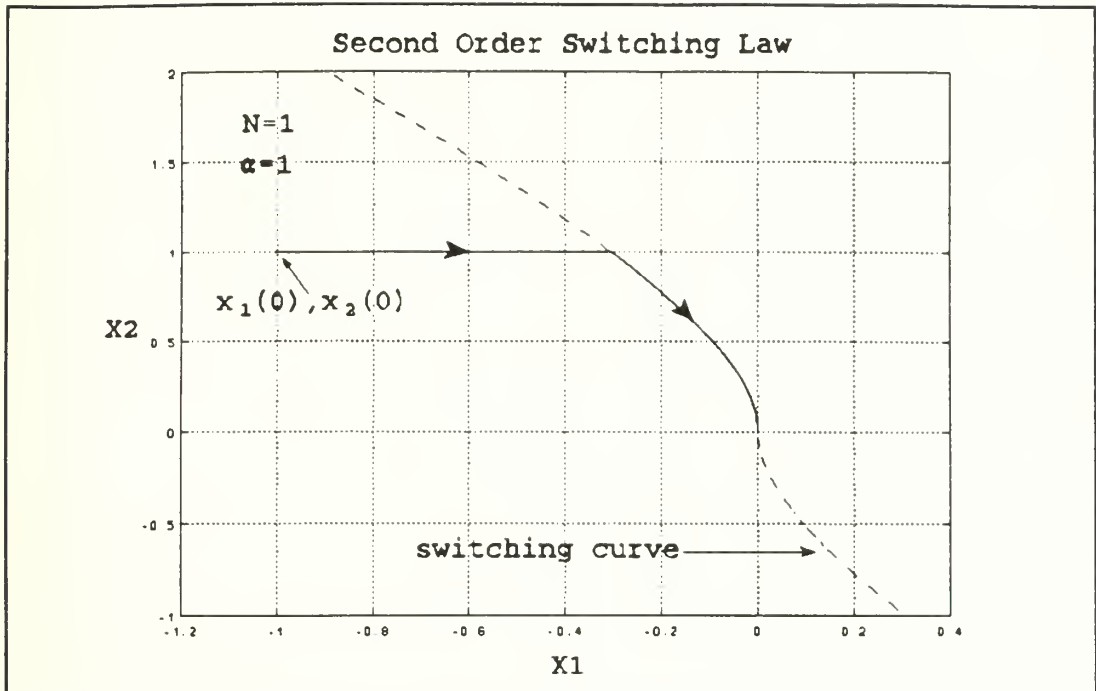


Figure 3.18 Simulation of Second Order Controller

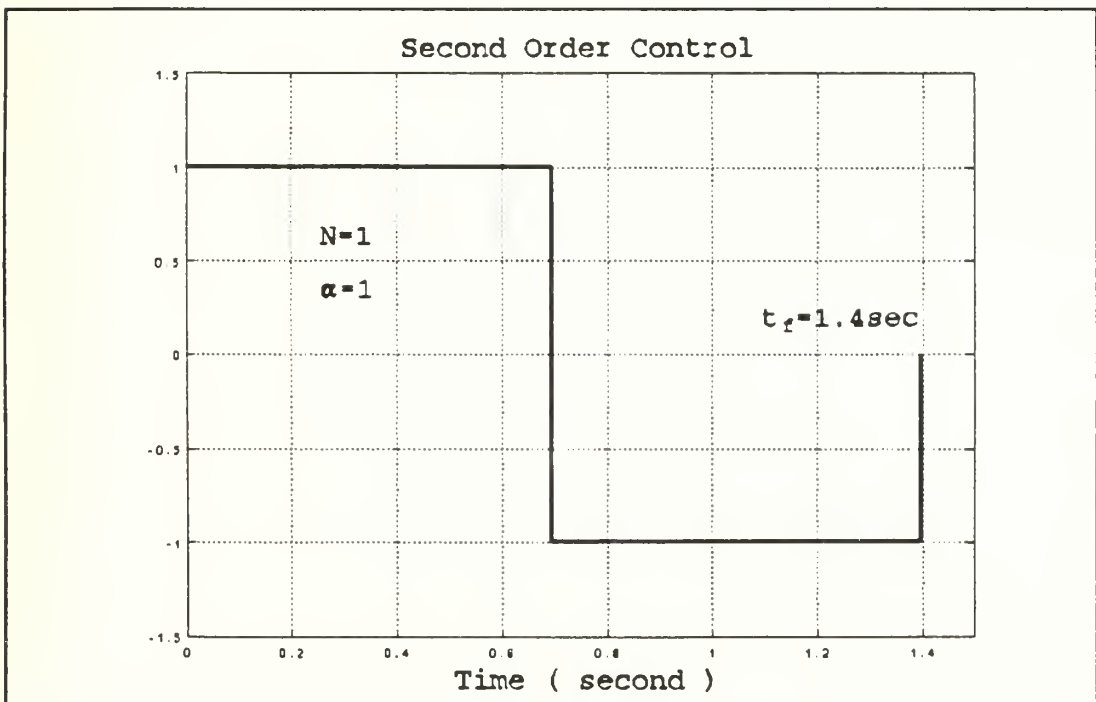


Figure 3.19 Control Effort for Second Order Controller

#### IV. THIRD ORDER CONTROLLER

In the previous chapter, we found the relations between the states in two-dimensional space. Now, we need to combine these relations in three-dimensional space in order to find the third order switching law.

##### A. FORWARD TIME SYSTEM

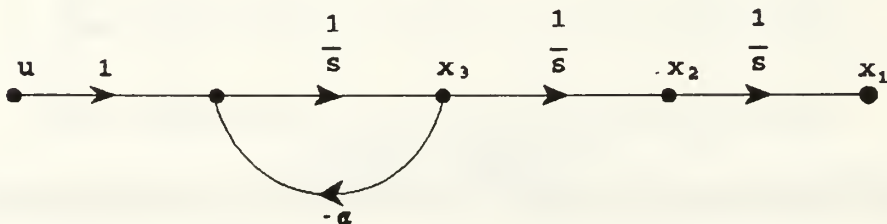
A third order minimum time controller can be used in a point defense system to position the missile in minimum time onto a "head-on" collision course with the target where the approaching missile is in its final trajectory and not maneuvering.

##### 1. System Definition

Our third order linear system is

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\alpha \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u \quad (4.1)$$

This can be represented in flow diagram as follows



The transfer function of the system is

$$\frac{X(s)}{U(s)} = G(s) = \frac{1}{s^2 (s + \alpha)} \quad (4.2)$$

Again, as the first step, we need to check the controllability and observability of the system.

$$Q = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -\alpha \\ 1 & -\alpha & \alpha^2 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (4.3)$$

Both matrices have a rank of 3 so that the system is controllable, observable. The eigenvalues of A, 0, 0 and  $-\alpha$ , are all nonpositive real, so an optimal control exists.

From the theorems due to Pontryagin, the control may change sign, at most, twice. Again going back in negative time, we may follow the zero trajectory curves out from the origin with control efforts of  $\pm N$ . Now, an infinite number of curves intersect these zero trajectory curves by making a surface. And, from this surface, an infinite number of trajectories take us to the initial conditions. So, we start with an arbitrary initial condition such that  $u = +N$  will drive the system to intersect with the surface as shown in Figure 4.1. On this surface, the control will switch to  $u = -N$  and drive the system along the surface to intersect with the zero trajectory curve. Here, the control will switch to  $u = +N$  again and will drive the system to the origin.



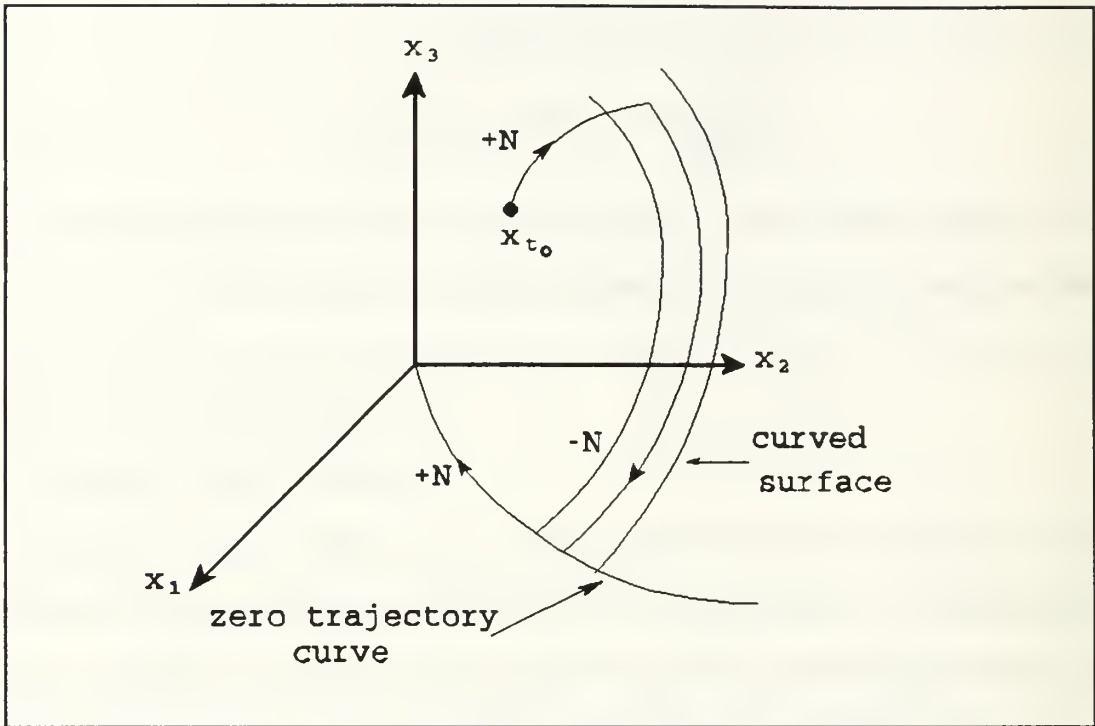


Figure 4.1 Three-Dimensional Switching Curves

## 2. Third Order Switching Curves

Starting with the state equation (4.1) and transforming this into the uncoupled system by equation (3.8) we get

$$G = \begin{bmatrix} \frac{1}{\alpha} & \frac{-1}{\alpha^2} & \frac{1}{\alpha^2} \\ 0 & \frac{1}{\alpha} & \frac{-1}{\alpha} \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad G^{-1} = \begin{bmatrix} \alpha & 1 & 0 \\ 0 & \alpha & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad (4.4)$$

$$\dot{y} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -\alpha \end{bmatrix} y + \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} u \quad (4.5)$$

or in scalar equations

$$\dot{y}_1(t) = y_2(0) \quad (4.6)$$

$$\dot{y}_2(t) = u(t) \quad (4.7)$$

$$\dot{y}_3(t) = -\alpha y_3(0) + u(t) \quad (4.8)$$

The Hamiltonian is

$$H = 1 + p_1 y_2 - p_3 \alpha y_3 + (p_2 + p_3) u \quad (4.9)$$

From Pontryagin [Ref.1] we can minimize  $J$  by minimizing the Hamiltonian. This is achieved with

$$u = -N \operatorname{sign}(p_2 + p_3) \quad (4.10)$$

where

$$\dot{p}_1(t) = -\frac{\partial H}{\partial y_1} = 0 \quad (4.11)$$

$$\dot{p}_2(t) = -\frac{\partial H}{\partial y_2} = -p_1(t) \quad (4.12)$$

$$\dot{p}_3(t) = -\frac{\partial H}{\partial y_3} = \alpha p_3(t) \quad (4.13)$$

so that

$$p_1(t) = p_1(0) \quad (4.14)$$

$$p_2(t) = p_2(0) - p_1(0) t \quad (4.15)$$

$$p_3(t) = p_3(0) e^{\alpha t} \quad (4.16)$$

The solution of state equation (4.5) is

$$y_1(t) = y_1(0) + y_2(0) t + \frac{1}{2} u t^2 \quad (4.17)$$

$$y_2(t) = y_2(0) + u t \quad (4.18)$$

$$y_3(t) = y_3(0) e^{-\alpha t} + \frac{u}{\alpha} (1 - e^{-\alpha t}) \quad (4.19)$$

When we discretize the equations

$$\Phi = \mathcal{L}^{-1}((sI - A)^{-1}) = \mathcal{L}^{-1} \begin{bmatrix} \frac{1}{s} & \frac{1}{s^2} & 0 \\ 0 & \frac{1}{s} & 0 \\ 0 & 0 & \frac{1}{s + \alpha} \end{bmatrix} = \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\alpha t} \end{bmatrix} \quad (4.20)$$

$$\Delta = \int_0^t \Phi B dt = \int_0^t \begin{bmatrix} t \\ 1 \\ e^{-\alpha t} \end{bmatrix} dt = \begin{bmatrix} \frac{1}{t^2} \\ t \\ -\frac{1}{\alpha} e^{-\alpha t} \end{bmatrix}_0^t = \begin{bmatrix} \frac{1}{2} t^2 \\ t \\ \frac{1}{\alpha} (1 - e^{-\alpha t}) \end{bmatrix} \quad (4.21)$$

Combining these we obtain

$$y(t) = \begin{bmatrix} 1 & t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{-\alpha t} \end{bmatrix} y(0) + \begin{bmatrix} \frac{1}{2} t^2 \\ t \\ \frac{1}{\alpha} (1 - e^{-\alpha t}) \end{bmatrix} u(0) \quad (4.22)$$

or in scalar equations

$$y_1(t) = y_1(0) + y_2(0) t + \frac{1}{2} u(0) t^2 \quad (4.23)$$

$$y_2(t) = y_2(0) + u(0) t \quad (4.24)$$

$$y_3(t) = y_3(0) e^{-\alpha t} + \frac{u(0)}{\alpha} (1 - e^{-\alpha t}) \quad (4.25)$$

## B. NEGATIVE TIME SYSTEM

In negative time, we start at the origin with a control until a switching point is determined to determine the other half of the boundary conditions. In negative time we have

$$\dot{y} = \begin{bmatrix} 0 & -1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha \end{bmatrix} y + \begin{bmatrix} 0 \\ -1 \\ -1 \end{bmatrix} u \quad (4.26)$$

Discretizing we find

$$\phi = \mathcal{L}^{-1}((sI - A)^{-1}) = \begin{bmatrix} 1 & -t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{\alpha t} \end{bmatrix} \quad (4.27)$$

and

$$\Delta = \int_0^t \phi B dt = \begin{bmatrix} \frac{1}{2} t^2 \\ -t \\ \frac{1}{\alpha} (1 - e^{\alpha t}) \end{bmatrix} \quad (4.28)$$

so that

$$y(t) = \begin{bmatrix} 1 & -t & 0 \\ 0 & 1 & 0 \\ 0 & 0 & e^{\alpha t} \end{bmatrix} y(0) + \begin{bmatrix} \frac{1}{2} t^2 \\ -t \\ \frac{1}{\alpha} (1 - e^{\alpha t}) \end{bmatrix} u(0) \quad (4.29)$$

or in scalar equations

$$y_1(t) = y_1(0) - y_2(0)t + \frac{1}{2}u(0)t^2 \quad (4.30)$$

$$y_2(t) = y_2(0) - u(0) t \quad (4.31)$$

$$y_3(t) = y_3(0) e^{\alpha t} + \frac{u(0)}{\alpha} (1 - e^{\alpha t}) \quad (4.32)$$

### 1. Solving For Negative Time Boundary Points

Since we are going back in negative time starting from the origin,  $y_1(0)=0$ ,  $y_2(0)=0$  and  $y_3(0)=0$ , we have

$$y_1(t) = \frac{1}{2} u(0) t^2 \quad (4.33)$$

$$y_2(t) = -u(0) t \quad (4.34)$$

$$y_3(t) = \frac{u(0)}{\alpha} (1 - e^{\alpha t}) \quad (4.35)$$

Setting  $u=+N$  and travelling back along the zero trajectory curve for 1 second, we find

$$y_1(1) = \frac{1}{2} N \quad (4.36)$$

$$y_2(1) = -N \quad (4.37)$$

$$y_3(1) = \frac{N}{\alpha} (1 - e^{\alpha}) \quad (4.38)$$

Similarly, we run the system in negative time for 2 and 3 seconds to obtain other boundary points as shown in Figure in 4.2, and listed in Table 1.

TABLE 1

NEGATIVE TIME BOUNDARY CONDITIONS

$u=+N$	$t=1$	$t=2$	$t=3$
$y_1(t)$	$\frac{1}{2}N$	$2N$	$\frac{9}{2}N$
$y_2(t)$	$-N$	$-2N$	$-3N$
$y_3(t)$	$\frac{N}{\alpha}(1-e^{\alpha})$	$\frac{N}{\alpha}(1-e^{2\alpha})$	$\frac{N}{\alpha}(1-e^{3\alpha})$

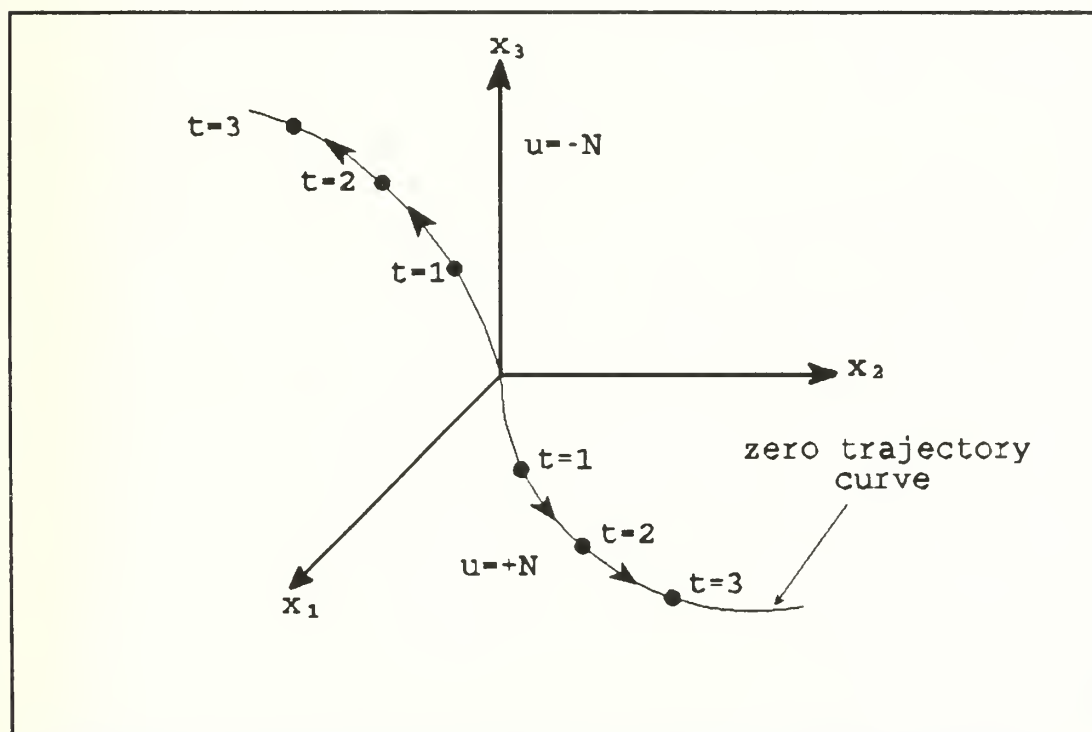


Figure 4.2 Boundary Values in Negative Time Solution

The control effort may also be  $u=-N$ . The boundary conditions in negative time along the zero trajectory curve with  $u=-N$  are listed in Table 2.

**TABLE 2**  
NEGATIVE TIME BOUNDARY CONDITIONS

$u=-N$	$t=1$	$t=2$	$t=3$
$y_1(t)$	$-\frac{1}{2}N$	$-2N$	$-\frac{9}{2}N$
$y_2(t)$	$N$	$2N$	$3N$
$y_3(t)$	$-\frac{N}{\alpha}(1-e^{\alpha})$	$-\frac{N}{\alpha}(1-e^{2\alpha})$	$-\frac{N}{\alpha}(1-e^{3\alpha})$

We may now solve for the family of curves that intersect the zero trajectory curves. Specifically, we solve for the equation of the curve that travels from some point  $y_1(0)$ ,  $y_2(0)$ ,  $y_3(0)$ , to the point at 1 second with the positions of the states given in Table 1. Since the control effort on the zero trajectory for this intersection point is  $u=+N$ , the control effort for the curve we are solving for must be  $u=-N$ . And the forward time equations must be equal to the negative time equations, so we may write

$$y_1(t) = \frac{1}{2}N = y_1(0) + y_2(0)t - \frac{1}{2}Nt^2 \quad (4.39)$$

$$y_2(t) = -N = y_2(0) - Nt \quad (4.40)$$

$$y_3(t) = \frac{N}{\alpha}(1-e^{\alpha}) = y_3(0)e^{-\alpha t} - \frac{N}{\alpha}(1-e^{-\alpha t}) \quad (4.41)$$



Solving equation (4.40) for  $t$  we find

$$t = 1 + \frac{y_2(0)}{N} \quad (4.42)$$

Substituting equation (4.42) into equations (4.39) and (4.41) we get

$$N = y_1(0) + \frac{y_2^2(0)}{2N} \quad (4.43)$$

$$0 = e^{-\alpha} e^{-\alpha \frac{y_2(0)}{N}} \left( y_3(0) + \frac{N}{\alpha} \right) - \frac{2N}{\alpha} + \frac{N}{\alpha} e^{\alpha} \quad (4.44)$$

We may generate a family of equations representing both sides of the zero trajectory curves, and different points of intersection along the curves which are listed in Table 3. The control effort,  $u$ , in Table 3, is the control effort for the zero trajectory that is intercepted. The time,  $t$ , is the time out from the origin to the intercept time for the negative system.

To combine all the equations from Table 3 into one solution we define

$$w = \text{sign} \left( y_1(0) + \frac{y_2(0) |y_2(0)|}{2N} \right) \quad (4.45)$$

$$f = y_1(0) + w \frac{y_2^2(0)}{2N} \quad (4.46)$$

and

$$z = \sqrt{\left| \frac{f}{N} \right|} \quad (4.47)$$

**TABLE 3**  
**FAMILY OF SOLUTION CURVES**

u=+N t=1	$N = y_1(0) + \frac{y_2^2(0)}{2N}$
	$0 = e^{-\alpha} e^{-\alpha \frac{y_2(0)}{N}} \left( y_3(0) + \frac{N}{\alpha} \right) - \frac{2N}{\alpha} + \frac{N}{\alpha} e^{\alpha}$
u=+N t=2	$4N = y_1(0) + \frac{y_2^2(0)}{2N}$
	$0 = e^{-2\alpha} e^{-\alpha \frac{y_2(0)}{N}} \left( y_3(0) + \frac{N}{\alpha} \right) - \frac{2N}{\alpha} + \frac{N}{\alpha} e^{2\alpha}$
u=+N t=3	$9N = y_1(0) + \frac{y_2^2(0)}{2N}$
	$0 = e^{-3\alpha} e^{-\alpha \frac{y_2(0)}{N}} \left( y_3(0) + \frac{N}{\alpha} \right) - \frac{2N}{\alpha} + \frac{N}{\alpha} e^{3\alpha}$
u=-N t=1	$-N = y_1(0) - \frac{y_2^2(0)}{2N}$
	$0 = e^{-\alpha} e^{\alpha \frac{y_2(0)}{N}} \left( y_3(0) - \frac{N}{\alpha} \right) + \frac{2N}{\alpha} - \frac{N}{\alpha} e^{\alpha}$
u=-N t=2	$-4N = y_1(0) - \frac{y_2^2(0)}{2N}$
	$0 = e^{-2\alpha} e^{\alpha \frac{y_2(0)}{N}} \left( y_3(0) - \frac{N}{\alpha} \right) + \frac{2N}{\alpha} - \frac{N}{\alpha} e^{2\alpha}$
u=-N t=3	$-9N = y_1(0) - \frac{y_2^2(0)}{2N}$
	$0 = e^{-3\alpha} e^{\alpha \frac{y_2(0)}{N}} \left( y_3(0) - \frac{N}{\alpha} \right) + \frac{2N}{\alpha} - \frac{N}{\alpha} e^{3\alpha}$

so that our family of curves is defined as

$$0 = \left( e^{-z\alpha} e^{-w \frac{\alpha y_2(0)}{N}} \right) \left( y_3(0) + w \frac{N}{\alpha} \right) + w \left( -\frac{2N}{\alpha} + \frac{e^{z\alpha} N}{\alpha} \right) \quad (4.48)$$

Equation (4.45) tells us which sign to apply depending on the direction of the zero trajectory curve. Equation (4.47) gives the magnitudes of the equation depending on the distance of the zero trajectory curve from the origin.

### C. THIRD ORDER SWITCHING LAW

Using equations (4.45), (4.46), (4.47), and (4.48) we can obtain the third order switching law. So, our third order switching law in the uncoupled space can be defined as

$$w = \text{sign} \left( y_1(0) + \frac{y_2(0) |y_2(0)|}{2N} \right) \quad (4.49)$$

$$f = y_1(0) + w \frac{y_2^2(0)}{2N} \quad (4.50)$$

$$z = \sqrt{\left| \frac{f}{N} \right|} \quad (4.51)$$

$$u = -N \text{sign} \left( e^{-z\alpha} e^{-w\alpha \frac{y_2(0)}{N}} \left( y_3(0) + w \frac{N}{\alpha} \right) + w \left( -\frac{2N}{\alpha} + \frac{e^{z\alpha} N}{\alpha} \right) \right) \quad (4.52)$$

We can now go back to our normal state space by using equation (3.11) so that

$$y_1 = \alpha x_1 + x_2 \quad (4.53)$$

$$y_2 = \alpha x_2 + x_3 \quad (4.54)$$

$$y_3 = x_3 \quad (4.55)$$

Therefore, the third order switching law is defined as

$$w = \text{sign} \left( \alpha x_1(0) + x_2(0) + \frac{(\alpha x_2(0) + x_3(0)) |\alpha x_2(0) + x_3(0)|}{2N} \right) \quad (4.56)$$

$$f = \alpha x_1(0) + x_2(0) + w \left( \frac{(\alpha x_2(0) + x_3(0))^2}{2N} \right) \quad (4.57)$$

$$z = \sqrt{\left| \frac{f}{N} \right|} \quad (4.58)$$

So finally, the switching law containing the coupled state values is:

$$u = -N \text{sign} \left( e^{-z\alpha} e^{-w\alpha \frac{(\alpha x_2(0) + x_3(0))}{N}} \left( x_3(0) + w \frac{N}{\alpha} \right) + w \left( \frac{-2N}{\alpha} + \frac{e^{z\alpha} N}{\alpha} \right) \right) \quad (4.59)$$

#### D. THIRD ORDER CONTROLLER SIMULATION

##### 1. Simulation in Uncoupled Space

A simulation of this regulator shows that the third order switching law of equation (4.49), drives the states first to the zero trajectory curve and then to the origin with 2 changes in the control effort. The first run is from the initial point of  $x_1(0) = -1/2$ ,  $x_2(0) = 0$ ,  $x_3(0) = 0$  with a maximum control effort of  $N=1$  and a time constant of  $\alpha=1$ . The results of this run are given in Figures 4.3 and 4.4. In order to show

the effects of maximum control effect,  $N$  and the time constant,  $\alpha$ , two other simulations are run. The results of these simulation are given in Figures 4.5, 4.6 and 4.7, 4.8.

## 2. Simulation of Minimum Time Control of the Third Order Regulator

Now, we need to simulate the regulator in 3 dimensional space. Again, a simulation of the third order regulator shows that the third order switching law of equation (4.56) drives the states to the origin with only 2 changes in the control effort. The results of this run, again with initial conditions of  $x_1(0)=-1/2$ ,  $x_2(0)=0$ ,  $x_3(0)=0$ , a maximum control effort of  $N=1$  and a time constant of  $\alpha=1$ , are given in Figures 4.9 and 4.10. With the same initial conditions in order to show the effects of  $N$  and  $\alpha$ , we run the system two more times. The results of the simulation with a control effort of  $N=5$ , and a time constant of  $\alpha=1$  are given in Figures 4.11 and 4.12. The results of the simulation with a control effort of  $N=1$  and a time constant of  $\alpha=5$  are given in Figures 4.13 and 4.14. Our switching law should work from an arbitrary initial condition. So, in order to test the switching law, we simulated the system from three different initial conditions. The first one is from the initial point of  $x_1=-1/2$ ,  $x_2=1$ ,  $x_3=1$  whose results are given in Figures 4.15 and 4.16. The second simulation is from  $x_1=0$ ,  $x_2=-1/2$ ,  $x_3=0$  whose results are given in Figures

4.17 and 4.18. And the last simulation is from the same initial point of  $x_1=0$ ,  $x_2=-1/2$ ,  $x_3=0$ , with a maximum control effort of  $N=5$  whose results are given in Figures (4.19) and 4.20.

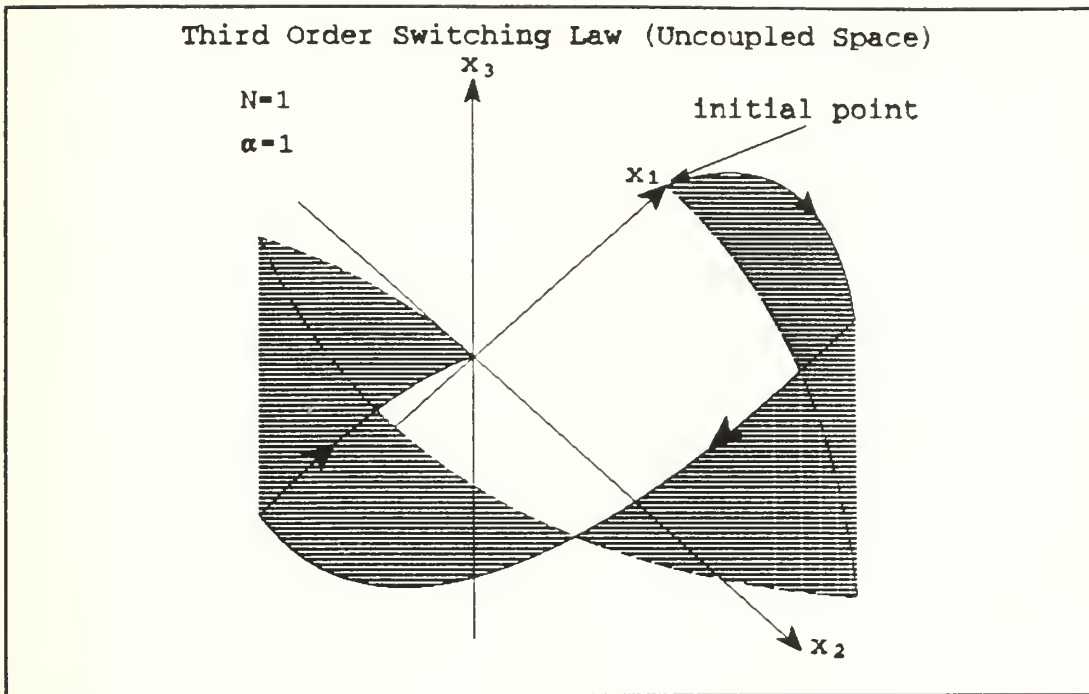


Figure 4.3 Simulation of Third Order Controller

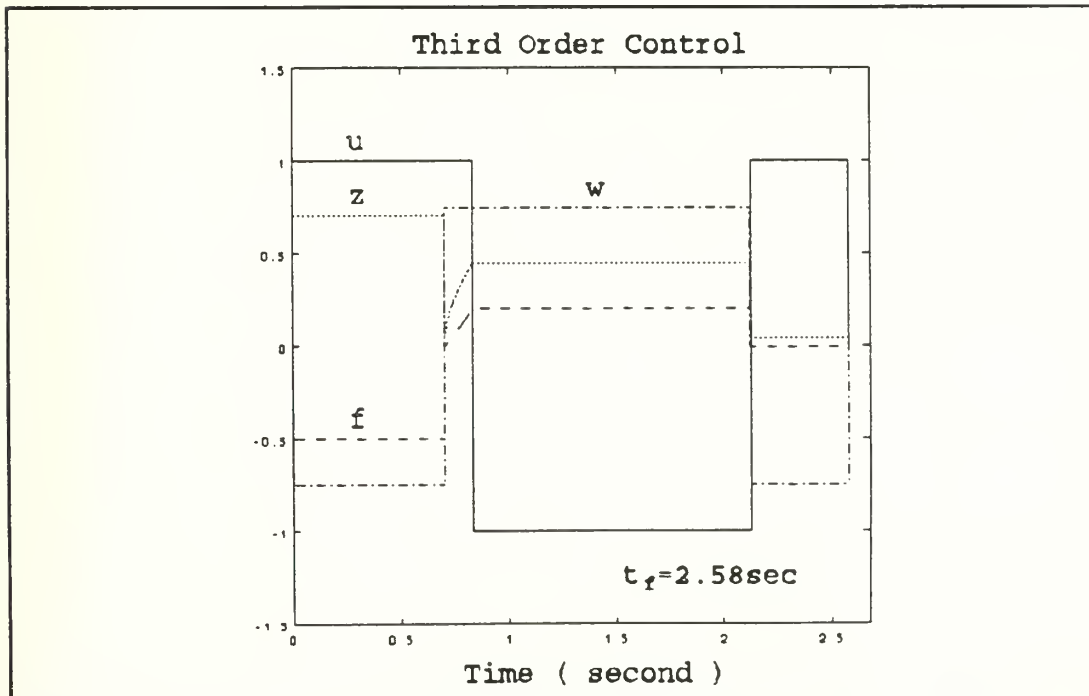


Figure 4.4 Control Effort for Third Order Controller



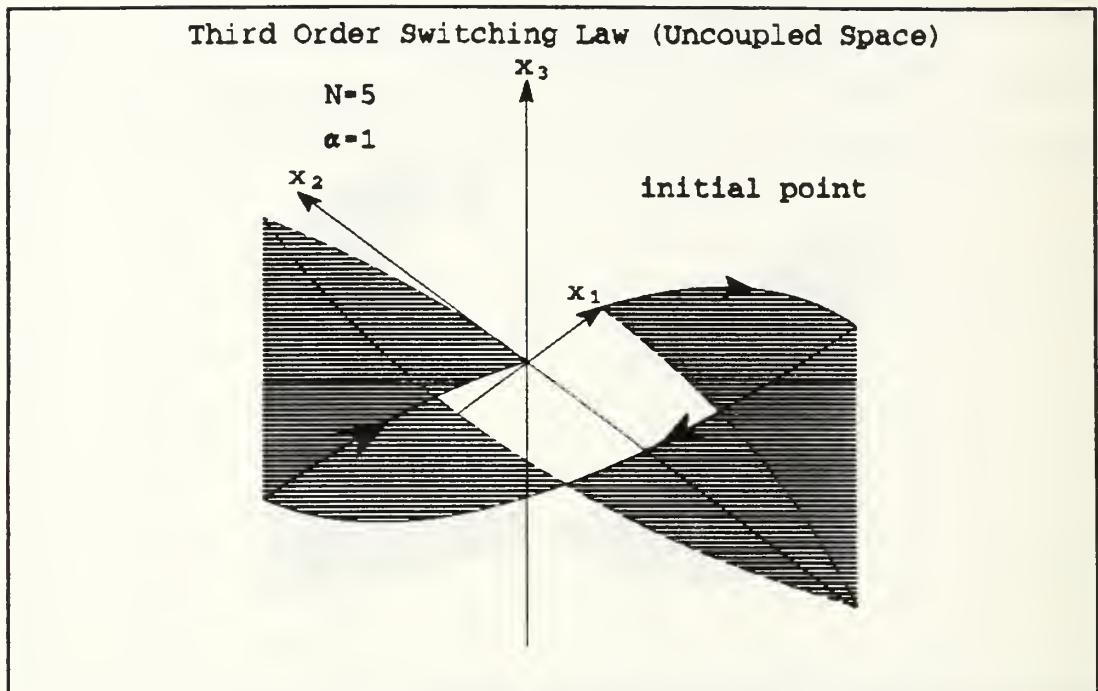


Figure 4.5 Simulation of Third Order Controller

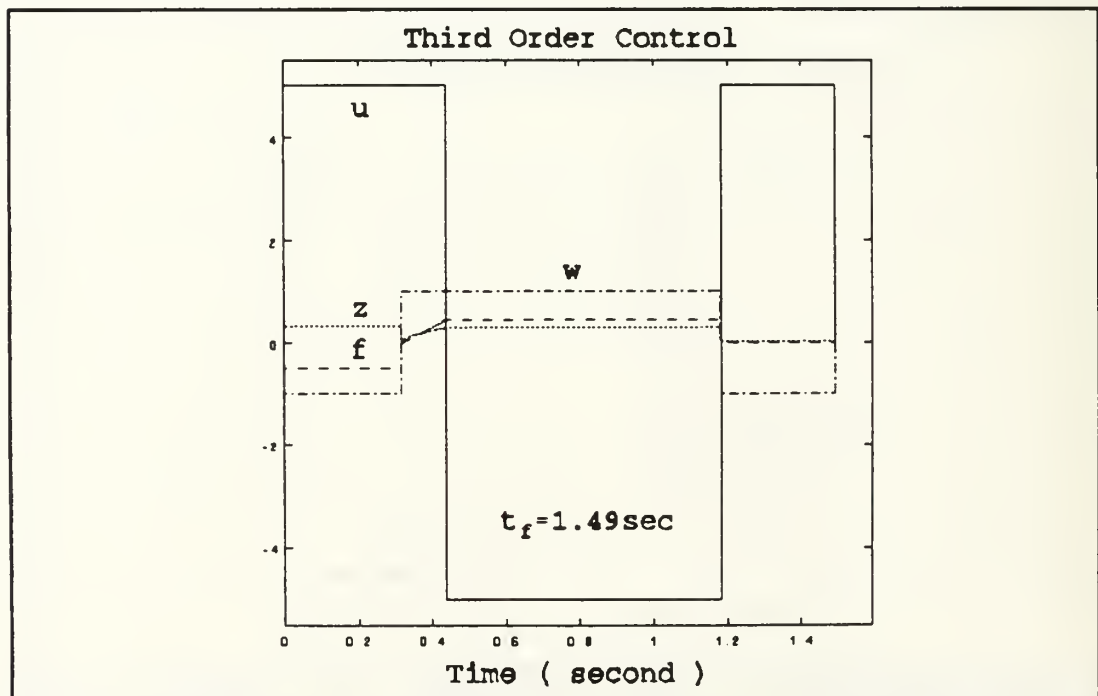


Figure 4.6 Control Effort for Third Order Controller

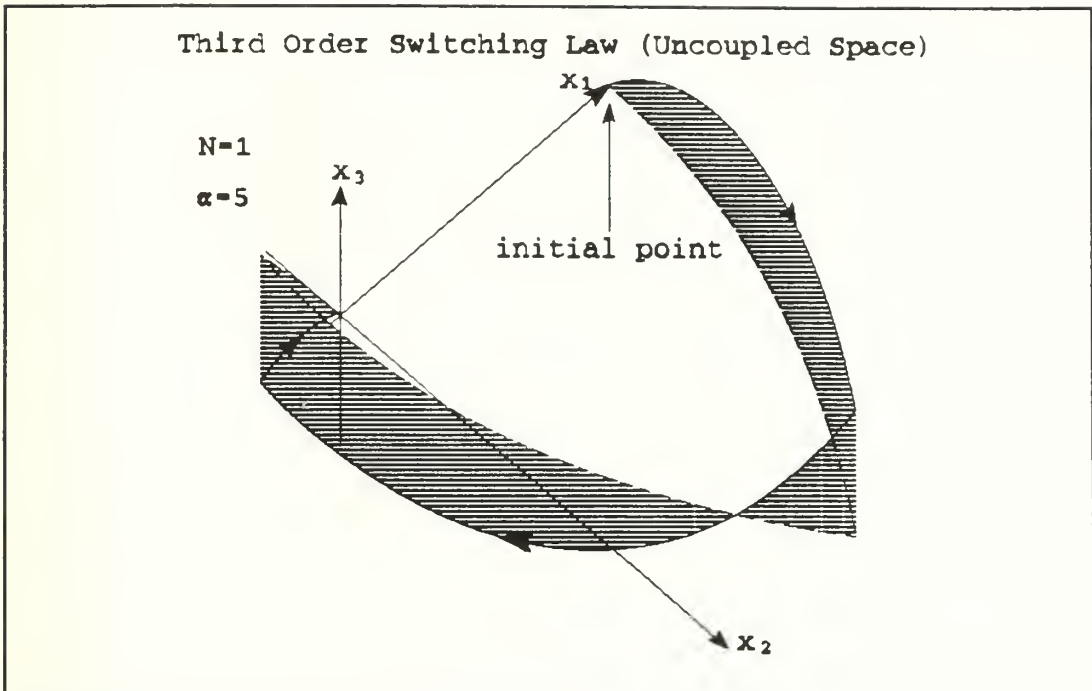


Figure 4.7 Simulation of Third Order Controller

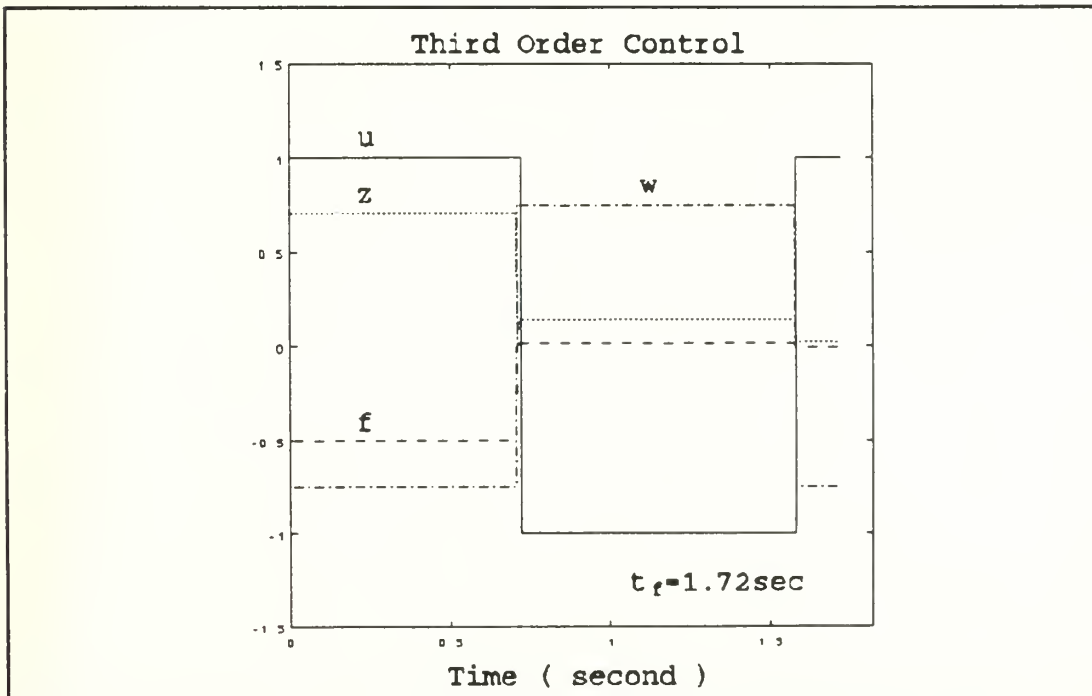


Figure 4.8 Control Effort for Third Order Controller

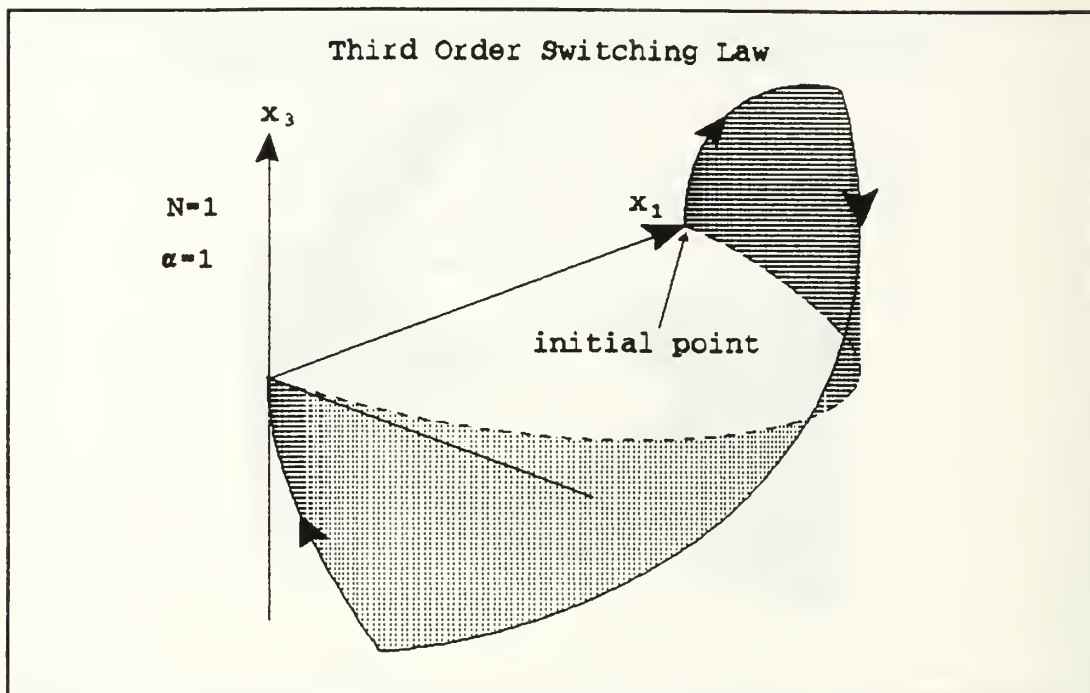


Figure 4.9 Simulation of Third Order Controller

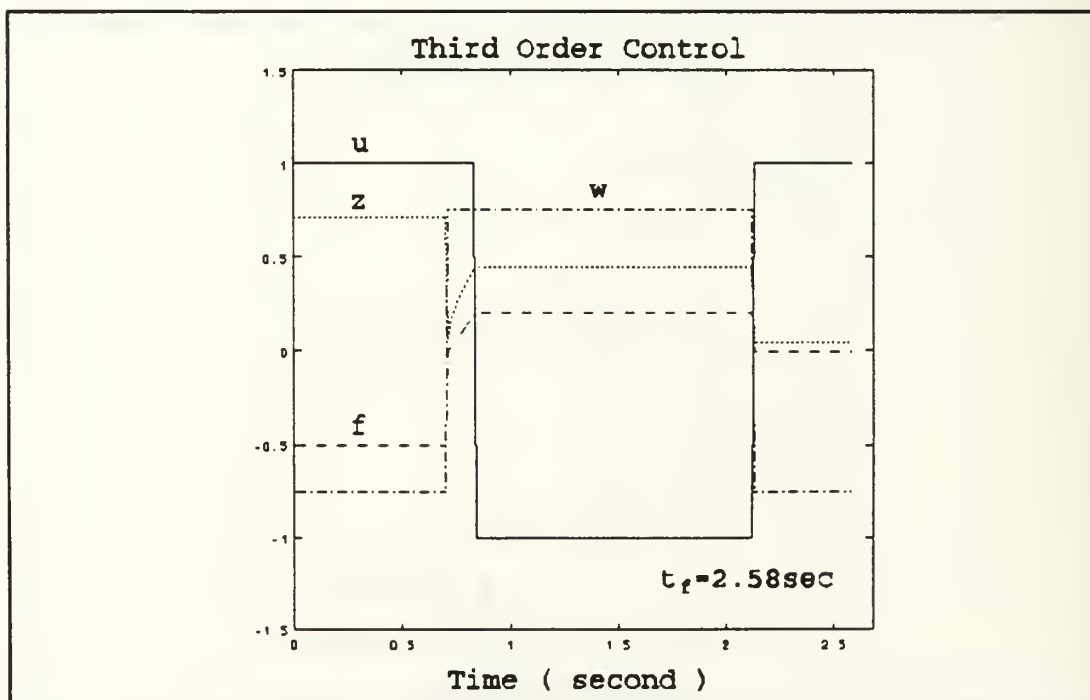


Figure 4.10 Control Effort for Third Order Control

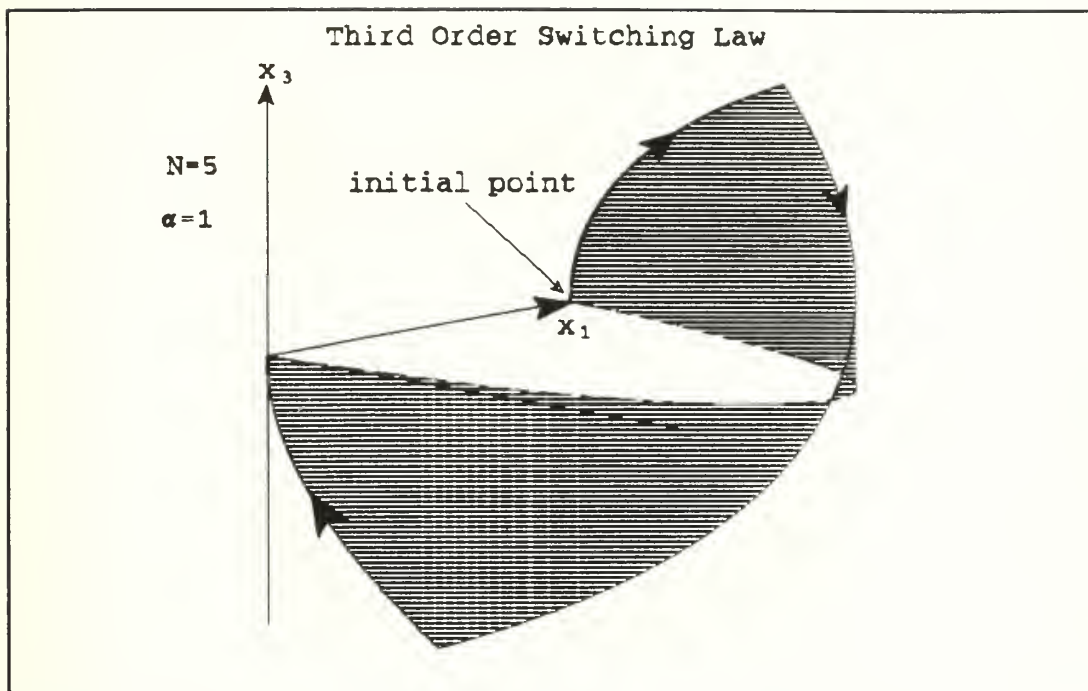


Figure 4.11 Simulation of Third Order Controller

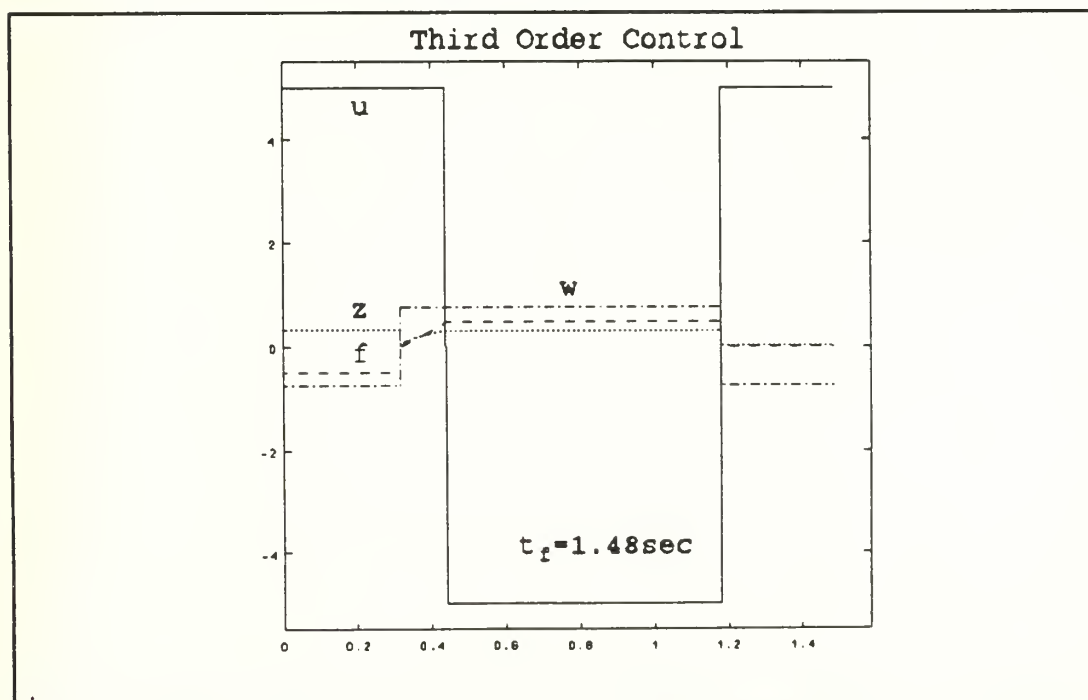


Figure 4.12 Control Effort for Third Order Controller

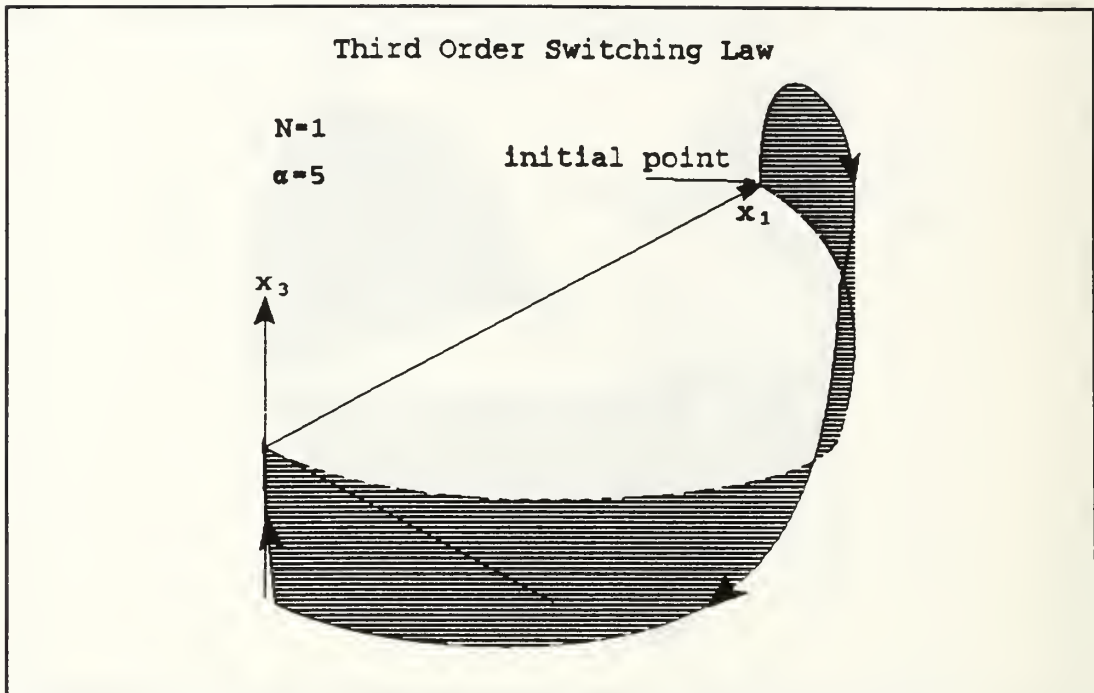


Figure 4.13 Simulation of Third Order Controller

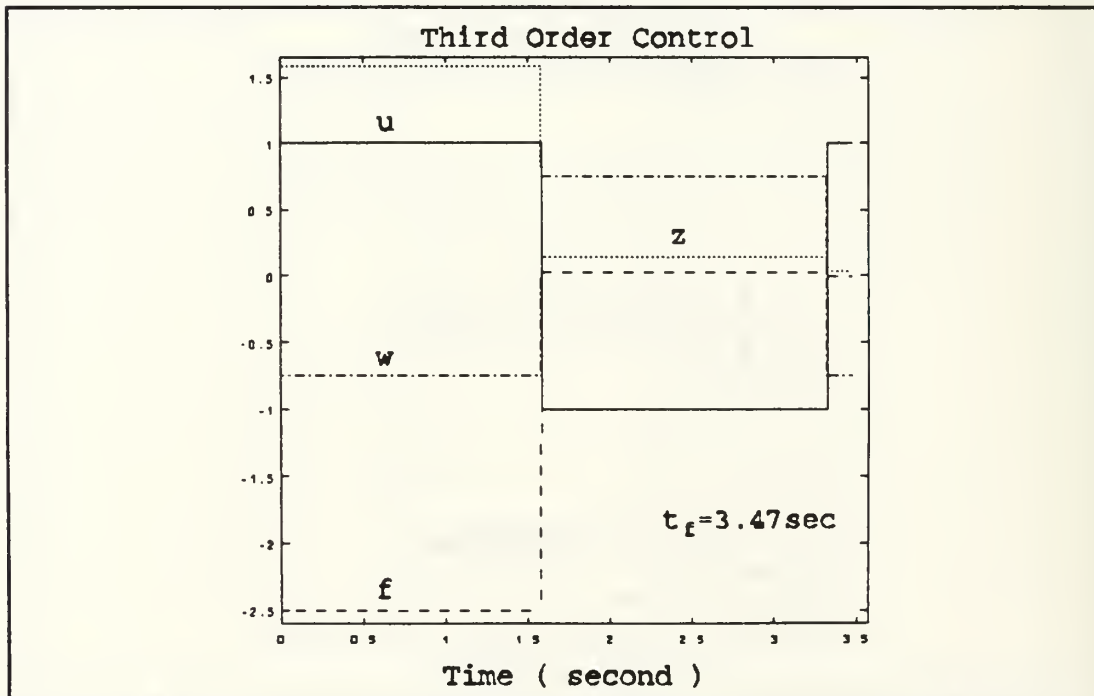


Figure 4.14 Control Effort for Third Order Controller

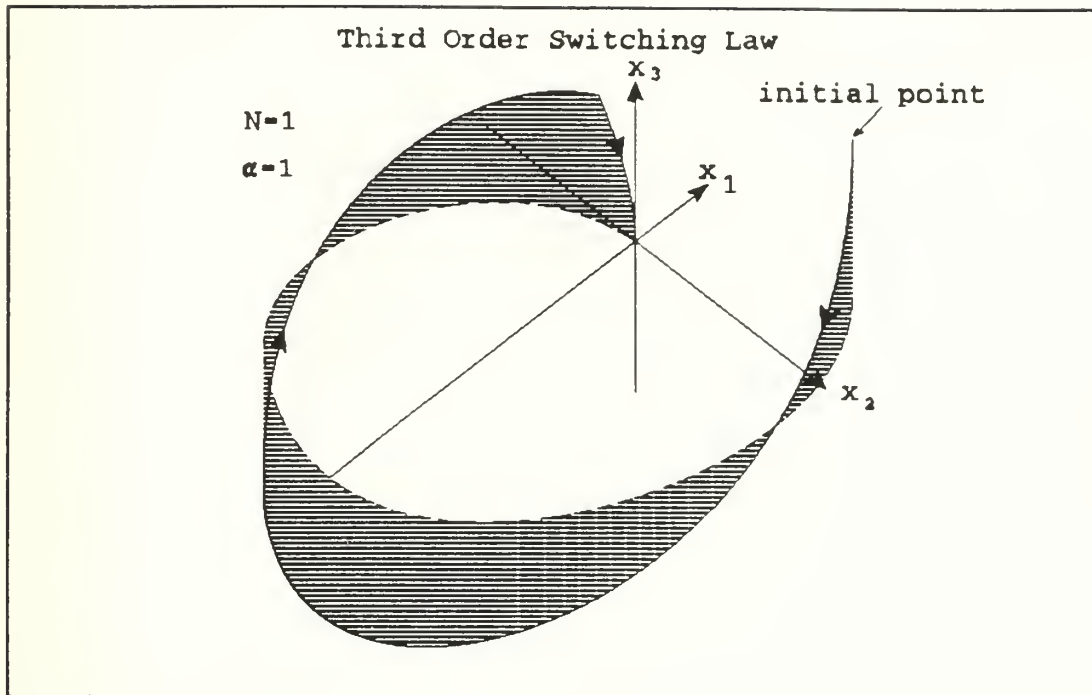


Figure 4.15 Simulation of Third Order Controller

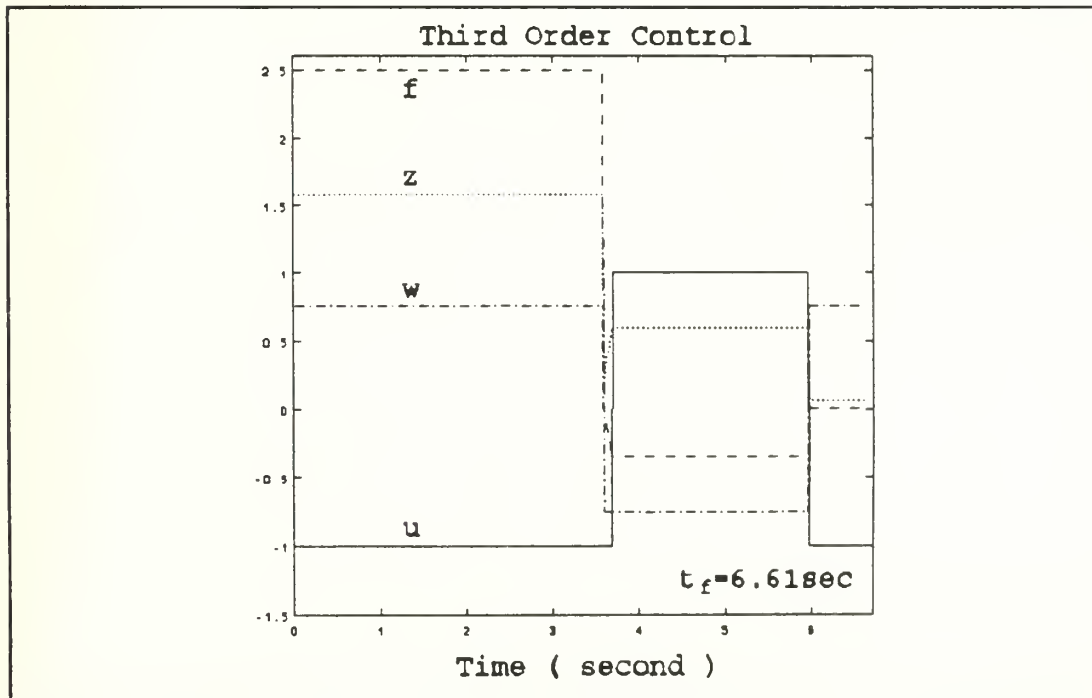


Figure 4.16 Control Effort for Third Order Controller

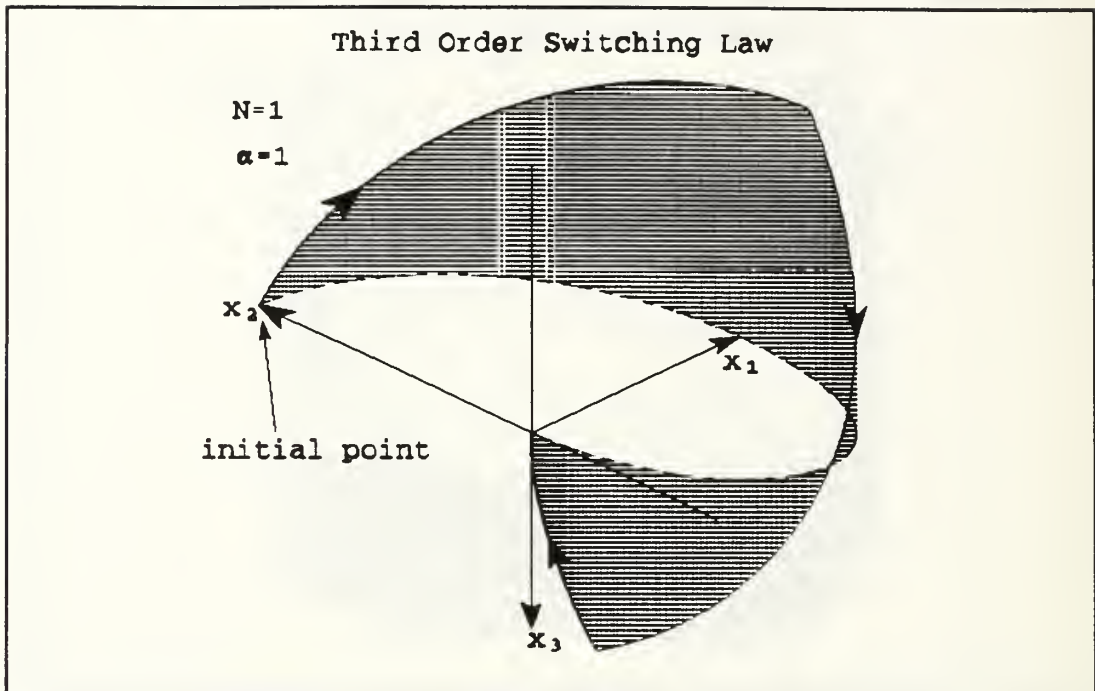


Figure 4.17 Simulation of Third Order Controller

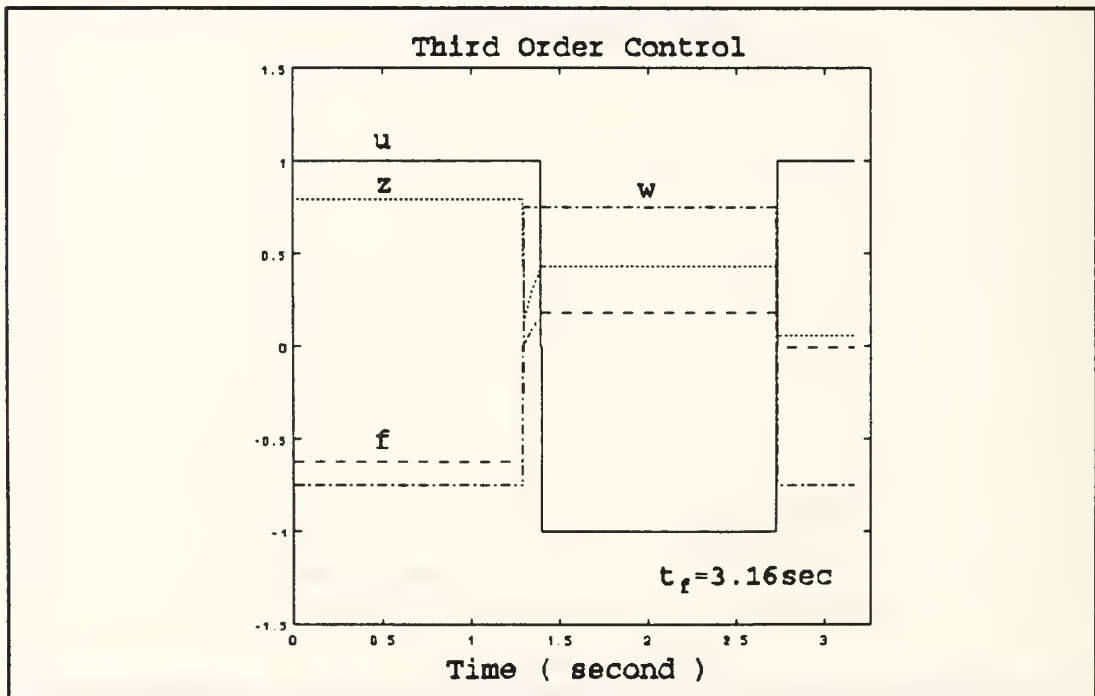


Figure 4.18 Control Effort for Third Order Controller



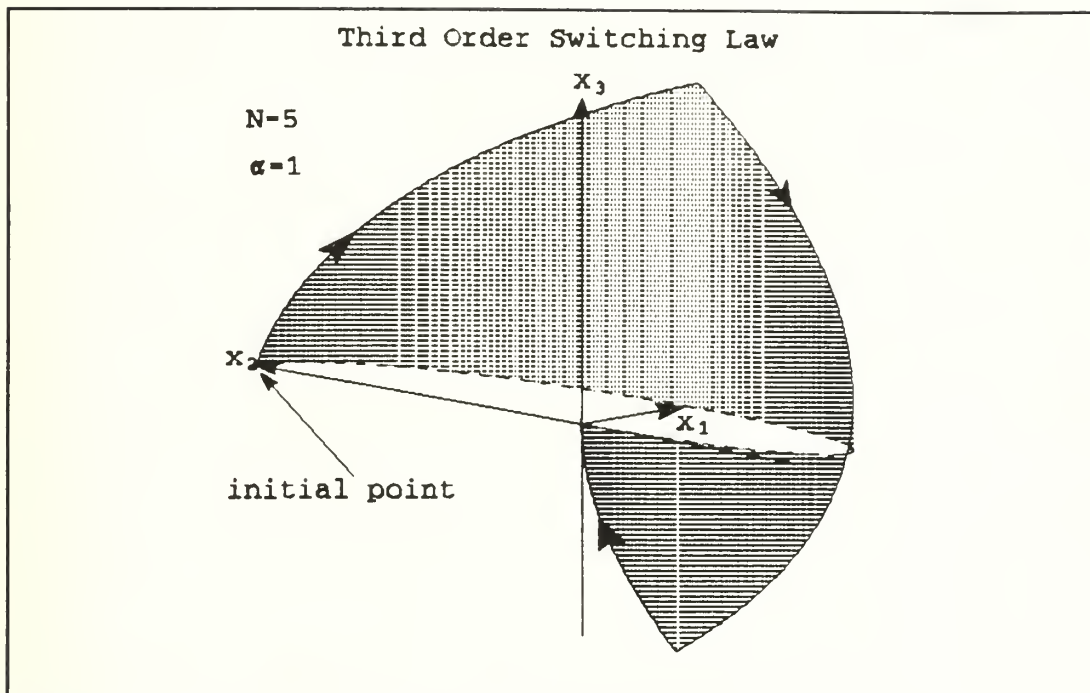


Figure 4.19 Simulation of Third Order Controller

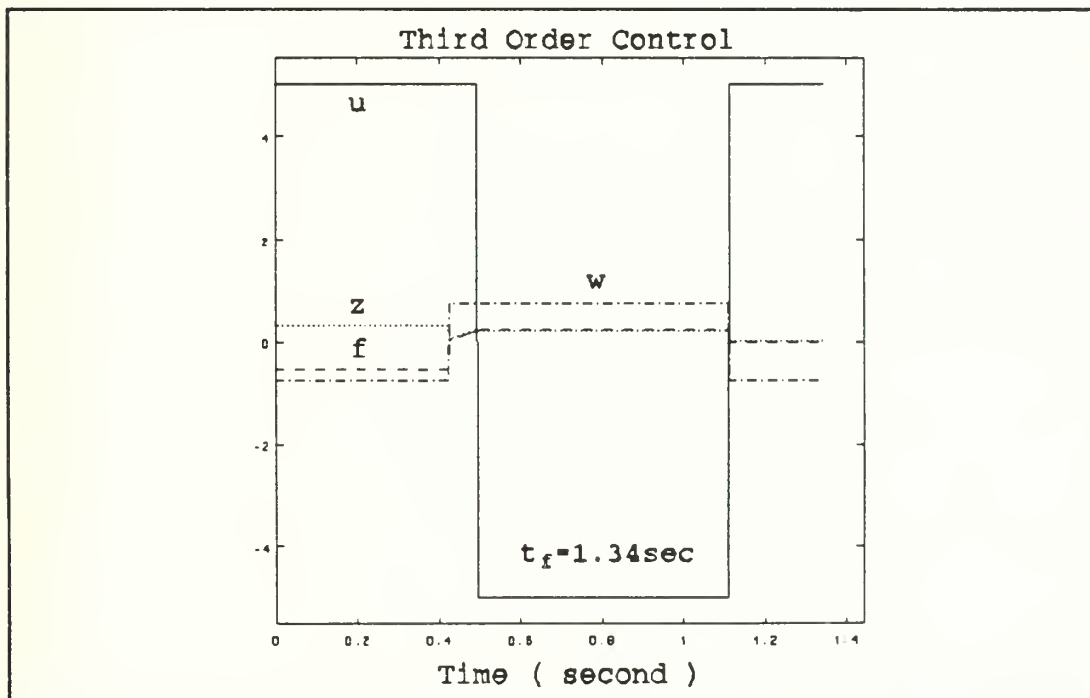


Figure 4.20 Control Effort for Third Order Controller

## V. CONCLUSION

We have developed a minimum time controller that drives the states to the origin for a third order regulator. Our controller has a maximum control effort,  $N$ , and a time constant,  $\alpha$ , so that both can be adjusted according to need.

As we can see from the simulations, when we increase the magnitude of the control effort, the response of the system is faster. The effect of the time constant is in the reverse direction. When we increase the magnitude of the time constant, the response of the system decreases.

After driving the states to the origin, the switching law may be confused at the origin, so chatter may occur. In order to prevent this, we may need to turn off the control effort upon reaching the origin. Increasing the sampling rate may decrease the magnitude of the chatter.

Our third order switching law can be applied to a fast reaction defense missile where the approaching missile has a speed advantage and the standard proportional navigation controller may not catch the approaching missile.

## APPENDIX. PROGRAM CODE

### 1. BB2NDUN.M

% BB2NDUN.M

11 October 1991

% This program is a simulation of the minimum time control of  
% the second order system in forward time in uncoupled space.

% written by Kayhan Vardareri

clc;clear

alpha =1;

N =1; % The magnitude of the control effort

A =[0 0;0 -alpha]; % State matrices of the system

B =[1;1];

C =[1 0];

Tf =4.5; % Length of simulation

dt =0.01; % Time increment for simulation

[phi,del] =c2d(A,B,dt); % Discretize the system

kmax =Tf/dt+1; % Maximum integer value for the simulation

y =zeros(2,kmax); % Storage vectors

d =zeros(1,kmax);

u =zeros(1,kmax);

time =zeros(1,kmax);

y10 =2; % Initial conditions for y

y20 =1;

```

y(:,1) =[y10;y20];      % Initial state vector
% Begin Simulation
for (i=1:kmax-1);
    u(1,i)=-N*sign(y(1,i)-sign(y(2,i))*N/alpha*log(...
        1+alpha/N*abs(y(2,i)))));
    y(:,i+1) =phi*y(:,i)+del*u(i);
    d =C*y(:,i+1);
    time(i+1) =time(i)+dt;
end
% Plots of The Outputs
plot(y(1,:),y(2,:));grid;
xlabel('Y1');ylabel('Y2');
title('Second Order Switching Law (Uncoupled Space)');
gtext(['N=',num2str(N)]);gtext(['alpha = ',num2str(alpha)]);
%meta kay2us
pause;clg;
plot(time,u)
xlabel('Time (sec)');ylabel('Magnitude');
title('Second Order Control');
%meta kay2uc

```

## 2. BB2ND.M

% BB2ND.M

11 October 1991

% This program is a simulation of the minimum time control of  
% the second order system in forward time.

% written by Kayhan Vardareri

clc;clear

alpha =1;

N =1; % The magnitude of the control effort

A =[0 1;0 -alpha]; % State matrices of the system

B =[0;1];

C =[1 0];

Tf =4.5; % Length of simulation

dt =0.01 % Time increment for simulation

[phi,del] =c2d(A,B,dt); % Discretize the system

kmax =Tf/dt+1; % Maximum integer value for the simulation

x =zeros(2,kmax); % Storage vectors

y =zeros(1,kmax);

u =zeros(1,kmax);

time =zeros(1,kmax);

x10 =1; % Initial conditions for x

x20 =1;

x(:,1) =[x10;x20]; % Initial state vector

% Begin Simulation

for (i=1:kmax-1);

```

u(i)=-N*sign(alpha*x(1,i)+x(2,i)-sign(x(2,i))...
      *(N/alpha*log(1+alpha/N*abs(x(2,i)))));
x(:,i+1) =phi*x(:,i)+del*u(i);
y =C*x(:,i+1);
time(i+1) =time(i)+dt;
end
% Plots of The Outputs
plot(x(1,:),x(2,:));grid;
xlabel('X1');ylabel('X2');
title('Second Order Switching Law');
gtext(['N = ',num2str(N)]);gtext(['alpha = ',num2str(alpha)]);
%meta kay2s
pause;clg;
plot(time,u)
xlabel('Time (sec)');ylabel('Magnitude');
title('Second Order Control');
%meta kay2c

```

### 3. BB3RDUN.M

```
% BB3RDUN.M                                11 October 1991
% This program is the simulation of the minimum time control
% of the third order regulator in uncoupled space.
% written by Kayhan Vardareri

clg;clear

alpha =1;

N =1;                                     % The magnitude of the control effort
A =[0 1 0;0 0 0;0 0 -alpha]; % State matrices of the system
B =[0;1;1];
C =[1 0 0];

Tf =2.578;                               % Length of simulation
dt =0.002;                               % Time increment for simulation
[phi,del] =c2d(A,B,dt); % Discretize the system
kmax =Tf/dt+1; % Maximum integer value for the simulation
y =zeros(3,kmax); % Storage vectors
d =zeros(1,kmax);
u =zeros(1,kmax);
w =zeros(1,kmax);
f =zeros(1,kmax);
z =zeros(1,kmax);
time =zeros(1,kmax);

y10 = -1/2; % Initial conditions for y
y20 = 0;
```



```

y30 = 0;
y(:,1) =[y10;y20;y30]; % Initial state vector
% Switching Law
for (i=1:kmax-1);
    w(i)=sign(y(1,i)+y(2,i)*abs(y(2,i))/(2*N));
    f(i)=y(1,i)+w(i)*(y(2,i)^2)/(2*N));
    z(i)=sqrt(abs(f(i)/N));
    u(i)=-N*sign(exp(-alpha*z(i))*exp(-w(i)*alpha...
        *y(2,i)/N)*(y(3,i)+w(i)*N/alpha)+w(i)*...
        ((-2*N/alpha)+(exp(z(i)*alpha)*N/alpha)));
% Begin Simulation
    y(:,i+1) =phi*y(:,i)+del*u(i);
    d =C*y(:,i+1);
    time(i+1)=time(i)+dt;
end
% Plots of The Outputs
plot3d(y(1,:),y(2,:),y(3,:),45,45);grid;
%meta kay3us
pause;clg;axis([0 Tf -1.5 1.5]);
plot(time,u,time,0,75*w,'-.',time,f,'*',time,z);
xlabel('Time (sec)');ylabel('Magnitude');
title('Third Order Control (Uncoupled Space)');
%meta kay3uc
axis;

```

#### 4. BB3RD.M

% BB3RD.M

11 October 1991

% This program is the simulation of the minimum time control  
% of the third order regulator in uncoupled space.

% written by Kayhan Vardareri

clc;clear

alpha =1;

N =1; % The magnitude of the control effort

A =[0 1 0;0 1 0;0 0 -alpha]; %State matrices of the system

B =[0;0;1];

C =[1 0 0];

Tf =2.578; % Length of simulation

dt =0.002; % Time increment for simulation

[phi,del] =c2d(A,B,dt); % Discretize the system

kmax =Tf/dt+1; % Maximum integer value for the simulation

x =zeros(2,kmax); % Storage vectors

y =zeros(1,kmax);

u =zeros(1,kmax);

w =zeros(1,kmax);

f =zeros(1,kmax);

z =zeros(1,kmax);

time =zeros(1,kmax);

x10 =-1/2; % Initial conditions for y

x20 = 0;

```

x30 = 0;
x(:,1) =[x10;x20;x30]; % Initial state vector
% Switching Law
for (i=1:kmax-1);
w(i)=sign(alpha*x(1,i)+x(2,i)+((alpha*x(2,i)+x(3,i))*...
    abs(alpha*x(2,i)+x(3,i))/(2*N)));
f(i)=alpha*x(1,i)+x(2,i)+w(i)*((alpha*x(2,i)+...
    x(3,i))^2)/(2*N));
z(i)=sqrt(abs(f(i)/N));
u(i)=-N*sign(exp(-alpha*z(i))*exp(-w(i)*alpha*(alpha*...
    x(2,i)+x(3,i))/N)*(x(3,i)+w(i)*N/alpha)+w(i)*...
    ((exp(alpha*z(i))*N/alpha)-(2*N/alpha))));
% Begin Simulation
x(:,i+1) =phi*x(:,i)+del*u(i);
y =C*x(:,i+1);
time(i+1)=time(i)+dt;
end
% Plots of The Outputs
plot3d(x(1,:),x(2,:),x(3,:),45,45);grid;
%meta kay3s
pause;clg;axis([0 Tf -1.5 1.5]);
plot(time,u,time,0,75*w,'-.',time,f,'*',time,z,'+');
xlabel('Time (sec)');ylabel('Magnitude');
title('Third Order Control');

```

%meta kay3c

axis

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- [2] Kirk, D.E, *Optimal Control Theory*, Prentice-Hall Inc, 1970.
- [3] Cooper, C.R., *Second and Third Order Minimum Time Controllers and Missile Adjoints*, MS Thesis, Naval Postgraduate School, June 1991.

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